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A SECOND ORDER NUMERICAL MODEL FOR HIGH VELOCITY IMPACT PHENOMENA

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A second order accurate code for penetration problems associated with high velocity impact of materials that exhibit elastic or elastic plastic behavior has been developed. The model uses a finite difference method based upon an Eulerian formulation of the basic differential equations. For interior mesh points; the difference equations are second order accurate. The boundary is explicity preserved as a discontinuity by the use of tracer particles whose positions mark the current interface or free surface position. On such curves free surface and interface boundary conditions are applied. An example of a

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I. Introduction

For many problems associated with the field of ballistic mechanics experimental methods required for model evaluation are very difficult to control in practice and are, in addition, very expensive to perform. Furthermore, it is very difficult, if not impossible, to obtain time dependent information experimentally on the state within solid materials. Thus, in many instances it is imperative that one use a numerical model which simulates the physical experiments to predict the required information.

The first step in the construction of the model is to define the governing differential equations. A more difficult task is to choose a numerical method which is used to replace the differential equations by a finite difference form; the resulting equations must be solved on a high speed digital computer. The main features that one strives for in the design of a numerical model are the attributes of accuracy, economy and ease of operation of the code for the designer-user. High velocity impact phenomena leads to very severe material distortions. The computer model presented in this report has been used to predict deformation states for problems in which the striking velocity lies in the range of 0.2-4.0 km/sec. For very much lower impact velocities computation times may become extreme; there does not appear to be a restriction on the method for higher impact velocities. As a result of difficulties associated with Lagrange methods when large distortions are present, it was felt that an Eulerian formulation was desirable.

Eulerian codes are characterized by a mesh which is fixed in space for all time; the material "flows" through this mesh. As such the Eulerian method has the intrinsic capability of representing numerical solutions to problems with large deformations over long periods of time without incurring the Lagrangian penalty of mesh distortion. As a result of maintaining the uniformity of the mesh the accuracy of the method is preserved and the need of operator intervention for rezoning the calculation is eliminated. In this paper we shall describe the key elements of an Eulerian code for ballistic problems which is second order accurate.

As is well known, and unless special care is taken, diffusion of one material into another material can occur in an Eulerian formulation. To prevent such diffusion between materials all moving surfaces which bound each domain are defined by material particles (called tracer particles) that are taken to be the end points of piecewise linear segments which approximate the boundary. These particles can move freely throughout the fixed Eulerian mesh; their motion is determined by ordinary differential equations. Although a Lagrangian calculation is necessarily introduced, the usual rezoning difficulties associated with such a formulation are simply solved since the problem is reduced to the redistribution of mesh points along a line which uses arc length as the independent variable. On this string one need not conserve quantities when rezoning is performed; indeed rezoning just requires maintaining uniform spacing of the particles. At free surfaces and interfaces the correct boundary conditions are imposed with no integration performed across boundaries separating materials.

The asymptotic accuracy of finite difference methods is measured by their order of accuracy. For a first order scheme the error is halved when the mesh spacing is halved. For a second order method halving the mesh spacing results in errors that are one quarter their previous value. Numerous computational experiments have verified that the extra accuracy associated with second order methods more than compensates for the additional work required. This accuracy can be utilized in two ways. One can fix the mesh spacings based on computer storage facilities and specific problem resolution requirements; the higher order methods will yield greater accuracy. Alternatively, one can fix the accuracy desired. In this case second order methods will need fewer zones compared to first order methods. As a result of the dependence of the time step on the mesh spacing, for explicit methods, less computer time is required for the specified accuracy.

In addition to being second order, the code that has been developed, called SMITE, uses a dissipative scheme in divergence form; hence shock waves imbedded within the flow region are treated automatically and with the correct jump conditions. This result is obtained because mass momentum and energy are conserved by the difference equations. Our model has been formulated in the cylindrical coordinates (z,r).

II. General Modeling Considerations

The differential equations describing the behavior of a continuous media involve both spatial and temporal derivatives of material properties. In order to compute the rate of change of certain of the variables one requires a knowledge of certain spatial derivatives.

The accuracy of the temporal changes depends not only on the accuracy of the difference scheme but also on the resolution of the mesh in the neighborhood of the regions of greatest variation in the solution. Such regions where functions can vary very rapidly on a length scale, which may be small compared to a characteristic length in the region where the differential equations are solved, is called a boundary layer.

It is obvious that imposing a fine mesh uniformly over the region of integration, so as to represent the function numerically by its values on the computational mesh, is an inefficient procedure. The fine grid spacing in the smooth region of the function is unnecessarily accurate and leads to large computational times.

Let r=a be a line in a region where the function is slowly varying while r=0 be a line in the boundary layer. Then the mesh spacing at r=0 should be less than the spacing at r=a by a factor d/a<1. One example used to achieve such spacing is a logarithmic variation of the mesh given by

$$\frac{\beta}{L} = 1 - \frac{\log\left(\frac{b+(a-r)}{b-(a-r)}\right)}{\log\left(\frac{b+a}{b-a}\right)} \quad , \; b = \frac{1}{1-\frac{d}{a}} \qquad (2.1)$$

Here L is a measure of the number of mesh points desired in the r direction and d is the length scale over which the solution exhibits its largest variation. The function $\beta(r)$ is the change of variables such that β is monotonic and most rapidly increasing in the thin boundary layer and more slowly increasing, with increasing r, in the region where the solution is smooth. With a suitable choice of β, such as the example (2.1), there will no longer appear in the functions which are approximated on the difference mesh, and which are considered to depend on the independent variable β, a boundary layer type of structure.

SMITE has, at present, a transformation for z which introduces the new coordinate α through

$$\alpha = D\,[z - g\cdot h(r)] \; ; \qquad (2.2)$$

this transformation has been used primarily for modeling conical shaped charges. The slope of a cone in the physical plane is q while h is used as a measure of the size of the cap at the vertex of the cone. In all subsequent discussion q and h are chosen such that $\alpha = Dz$ is the transformation with D a conversion for centimeters into coordinate lines. The choice of parameters used in transformations (2.1) and (2.2) is described in Section VI.

The basic equations of hydrodynamics require, at each time step, the evaluation of the divergence of the flux of the conserved physical quantities, i.e. $\frac{\partial f}{\partial z}$ and $\frac{\partial g}{\partial r}$. Under the condition of coordinate substitution mentioned above in which $\beta=\beta(r)$, i.e. Equation (2.1), the divergence of the flux becomes

$$g_r = \frac{\partial g}{\partial r} = \frac{\partial g}{\partial \beta}\frac{\partial \beta}{\partial r} \quad , \quad f_z = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial z}$$

We find that, in the case of (2 1),

$$\frac{\partial \beta}{\partial r} = \frac{d\beta}{dr} = \frac{2Lb}{\log \frac{b+a}{b-a}} \frac{1}{b^2-(a-r)^2}$$

which shows that the mesh spacing in cylindrical coordinates is proportional to $b^2-(a-r)^2$. Hence, as r varies from zero to a the mesh spacing varies from b^2-a^2, the smallest variation to b^2 which is the largest variation.

The introduction of such coordinate transformations introduces the concept of a computational plane (coordinates α-β) in which the grid is uniform but whose image in the physical plane (coordinate z-r) is dense in the boundary layer region and gets progressively sparse in regions where gradients in the solution are small. This results in economical use of the computing time.

Through application of the chain rule, it is possible to transform the divergence free equations from the physical space z-r to the computational plane α-β in such a manner that the new equations are still in divergence free form. A direct consequence is that internal shocks will be computed with the correct jump conditions in the computational plane.

The difference schemes that we use to approximate the divergence form of the differential equations requires a nine point rectangular lattice on the initial data plane at time level, t. Symmetrically placed about the point (i,j), at which we wish to find the functions which constitute the solution, are eight nearest neighbors whose position is defined by translations about ($i\Delta z$, $j\Delta r$) by $\pm\Delta z$ and $\pm\Delta r$. Nine point stencils which are entirely inside the domain of integration at time t are updated to time $t+\Delta t$ on the basis of a two step difference algorithm (see Section 4).

For mesh points near boundaries the nine point stencil required by the two step method is no longer completely contained within the domain of integration. In this case we no longer consider the divergence form for the differential equations but instead solve a quasilinear system of equations derived from the conservation equations for the unknown functions. The solution at these points is updated by a non-linear version of the Lax-Wendroff method. This method allows one to take into account in the difference equations the non-centered spacing required as a result of the boundary crossing between adjacent points of the stencil. This non-centering occurs in the difference formulas since data along the boundary must now be used instead of data at regular mesh points. As any of the eight nearest neighbors can be excised from the stencil by the boundary, there exists 256 possible ways of updating the solution at these "irregular mesh points". Since many of these truncated stencils are mirror images of each other, the required logic can be minimized.

In order to calculate the non-centered space differences to the first and second derivatives, we must know the position of the boundary together with the value of the dependent variables at all crossings of the boundary with coordinate mesh lines. This is accomplished by interpolation from the material particles defining the boundary, care being taken to account for possible multiple crossings of the boundary with a given coordinate line. Once all spatial differences are known to second order the solution at time $t+\Delta t$ is calculated by a truncated Taylor Series for the unknown vector function $W(t+\Delta t)$ about time t, keeping all terms through second order.

The moving boundaries consist of segments which satisfy either free surface conditions or contact discontinuity conditions. In addition, there may exist fixed boundaries some of which are lines of symmetry; here we use the usual reflection conditions to advance the solution. At such fixed boundaries the solution is reflected across the boundary to image points which allow the use of regular difference equations on the boundary. All surfaces, which move through the Eulerian mesh, are marked by material particles which obtain velocity components from the interior of the domain; their motion is governed by Lagrangian equations which are integrated at each time step to predict the new boundary position.

In the next sections we make a more detailed discussion of the above general considerations.

III. Basic Differential Equations

a) Axisymmetric Elastic Model

When a material supports shear stresses, it is necessary to include, in addition to the pressure forces, terms which account for the presence of these stresses. The equations of motion for such a material can be derived by applying the physical laws describing the conservation of mass, momentum and energy to a finite element of the material body. In addition, a statement of the stress-strain relationship of the material is required. For this paper a linear theory is assumed, i.e. material bodies will satisfy Hooke's law. Then these laws may be usefully written as a set of partial differential equations in a cylindrical coordinate system, as follows. If the substantial or particle derivative is defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial z} + v\frac{\partial}{\partial r}$$

then the conservation of mass can be written in terms of the density (the mass per unit volume) ρ, and the divergence of the velocity field, with components u and v in the z and r direction respectively, as

$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) \qquad (3.a.1)$$

The two momentum laws reflect the appearance of the stress components τ_{ij}. The axial momentum equation is

$$\rho\frac{du}{dt} = \frac{\partial\tau_{11}}{\partial z} + \frac{\partial\tau_{12}}{\partial r} + \frac{\tau_{12}}{r} \qquad (3.a.2)$$

and the radial momentum equation is

$$\rho\frac{dv}{dt} = \frac{\partial\tau_{12}}{\partial z} + \frac{\partial\tau_{22}}{\partial r} + \frac{\tau_{22}-\tau_{33}}{r} \qquad (3.a.3)$$

The evolution equation for the internal energy, e, per unit volume is given by

$$\rho\frac{de}{dt} = \tau_{11}\frac{\partial u}{\partial z} + \tau_{22}\frac{\partial v}{\partial r} + \tau_{12}\left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\right) - \frac{v(\tau_{11}+\tau_{22}+3p)}{r} \qquad (3.a.4)$$

The stresses required in the above relations must be obtained from the strains and strain rates. The linear stress-strain laws, with correction for rotation, are usually written in terms of deviator stresses S_{ij}. The rate of change of the stress component S_{ij} are given in terms of the strain rate, $\dot{e}_{ij}$, via

$$\frac{dS_{11}}{dt} = \frac{2\mu}{3}\left(2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} - \frac{v}{r}\right) + \tau_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \qquad (3.a.5)$$

$$\frac{dS_{12}}{dt} = \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) - \frac{S_{11}-S_{22}}{2}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \qquad (3.a.6)$$

$$\frac{dS_{22}}{dt} = \frac{2\mu}{3}\left(2\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z} - \frac{v}{r}\right) - \tau_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \qquad (3.a.7)$$

$$\frac{dS_{33}}{dt} = -\frac{2\mu}{3}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} - \frac{2v}{r}\right) \qquad (3.a.8)$$

The above Hookian laws, Equations (3.a.5)-(3.a.8), are connected to the evolution laws (3.a.1)-(3.a.4) by the algebraic conditions

$$\tau_{ij} = S_{ij} - p\delta_{ij} \qquad \delta_{ij} = 1 \text{ for } i = j$$
$$= 0 \text{ otherwise} \qquad (3.a.9\text{-}3.a.11)$$

The pressure p is related to the density ρ and specific internal energy e through the equation of state

$$p = P(\rho, e) \qquad (3.a.12)$$

The above set form a system of twelve equations for the twelve unknowns $\rho, u, v, e, p, \tau_{11}, \tau_{12}, \tau_{22}, \tau_{33}, S_{11}, S_{22}$ and S_{33}.

At this point we show that Equations (3.a.1)-(3.a.12) form a system which is not self consistent. To see this add Equations (3.a.5), (3.a.7) and (3.a.8). It is clear then that the sum $\sum_{i=1}^{3} \dot{S}_{ii}$ satisfies

$$\frac{d}{dt}(S_{11} + S_{22} + S_{33}) = 0 \qquad (3.a.13)$$

which implies that the sum of these stress deviators is a constant of the motion of the material; without loss of generality this constant can be taken to be zero for at t=0 each $S_{ii}=0$. Thus

$$\sum_{i=1}^{3} S_{ii} = 0 \qquad (3.a.14)$$

for all time.

Now if we sum Equations (3.a.9) through (3.a.11) for i=j, we obtain the relation

$$\tau_{11} + \tau_{22} + \tau_{33} = S_{11} + S_{22} + S_{33} - 3p \qquad (3.a.15)$$

which yields, after satisfying Equation (3.a.14),

$$\tau_{11} + \tau_{22} + \tau_{33} = - 3p \qquad (3.a.16)$$

Equation (3.a.16) states that the pressure p is determined by the mean of the stress tensor. This is a contradiction of Equation (3.a.12) which states the pressure is a function only of the density and internal energy.

Hooke's laws can be written in the form

$$\tau_{ii} = 2\mu e_{ii} + \lambda \sum_j e_{jj} \qquad i = 1,2,3 \qquad (3.a.17)$$

$$\tau_{ij} = 2\mu e_{ij}$$

Here μ is the shear modulus of the material, λ is a Lamé constant and the strain e_{ij} is defined by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial \bar{x}_i}{\partial x_j} + \frac{\partial \bar{x}_j}{\partial x_i}\right) \qquad (3.a.18)$$

The displacements are $\bar{x}_i$.

Differentiating Equations (3.a.17) and (3.a.18) with respect to time yields

$$\dot{\tau}_{ii} = 2\mu\dot{e}_{ii} + \lambda\sum_j \dot{e}_{jj}$$

$$\dot{\tau}_{ij} = 2\mu\dot{e}_{ij} \tag{3.a.19}$$

The corresponding strain rate tensor is then given in terms of the velocity gradient,

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

with u_i and u_j the components of the velocity.

If Equation (3.a.16) is differentiated with respect to time we can compute p; using Equation (3.a.19) for $\dot{\tau}_{ii}$ we obtain:

$$\begin{aligned}\dot{p} &= -\frac{1}{3}\Sigma\dot{\tau}_{ii} \\ &= -\frac{1}{3}\left[2\mu\Sigma\dot{e}_{ii} + 3\lambda\Sigma\dot{e}_{ii}\right] \\ &= -\frac{2\mu+3\lambda}{3}\Sigma\dot{e}_{ii} = -k\,\mathrm{div}\,\bar{u} \\ &= -k\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right)\end{aligned} \tag{3.a.20}$$

We have defined the constant k, the bulk modulus, in terms of the Lamé constants μ and λ.

Clearly Equation (3.a.20), obtained from Hooke's law is not consistent with Equation (3.a.12). In fact Equation (3.a.20) together with Equation (3.a.1) yields

$$\frac{dp}{dt} = \frac{k}{\rho}\frac{d\rho}{dt}$$

which can be integrated from the lower limit $(p,\rho)=(0,\rho_0)$ to the state (p,ρ); this yields for the pressure

$$p = k\mathrm{Ln}(1+\mu) \quad , \mu = \frac{\rho}{\rho_0} - 1 \tag{3.a.21}$$

Wilkins[1] points out that Equation (3.a.21) can be expanded in a Taylor series for $\rho/\rho_0 \sim 1$ to obtain a polynomial for p as a function of μ. Thus, Equation (3.a.12) should, for consistency, be close to Equation (3.a.21) but obviously cannot in general be the same. To be consistent then, there is really no degree of freedom in the choice of the form for an equation of state Equation (3.a.12) in the system (3.a.1)-(3.a.12).

One possible way to avoid the above inconsistency is to replace the stresses τ_{ij} appearing in Equations (3.a.1)-(3.a.4) by the deviatoric stresses S_{ij} and the pressure p through the use of Equations (3.a.9)-(3.a.11) and then eliminate Equations (3.a.9)-(3.a.11). The stress deviator S_{33} can also be eliminated by using Equation (3.a.14). The resulting set of equations can be written as

$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) \tag{3.a.22}$$

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial z} + \frac{\partial S_{11}}{\partial z} + \frac{\partial S_{12}}{\partial r} + \frac{S_{12}}{r} \tag{3.a.23}$$

$$\rho\frac{dv}{dt} = -\frac{\partial p}{\partial r} + \frac{\partial S_{12}}{\partial z} + \frac{\partial S_{22}}{\partial r} + \frac{2S_{22}+S_{11}}{r} \tag{3.a.24}$$

$$\rho\frac{de}{dt} = (S_{11}-p)\frac{\partial u}{\partial z} + (S_{22}-p)\frac{\partial v}{\partial r} + S_{12}\left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\right) - \frac{v(S_{11}+S_{22}+p)}{r} \tag{3.a.25}$$

$$\frac{dS_{11}}{dt} = \frac{2\mu}{3}\left(2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} - \frac{v}{r}\right) + S_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \tag{3.a.26}$$

$$\frac{dS_{12}}{dt} = \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) - \frac{S_{11}-S_{22}}{2}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \tag{3.a.27}$$

$$\frac{dS_{22}}{dt} = \frac{2\mu}{3}\left(2\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z} - \frac{v}{r}\right) - S_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \tag{3.a.28}$$

$$p = P(\rho, e) \tag{3.a.29}$$

We now have eight equations for the eight unknowns $\rho, u, v, e, S_{11}, S_{12}, S_{22}$ and p. Equations (3.a.22)-(3.a.29) are consistent. The only disadvantage of this set of equations is that Hooke's law of linear elasticity can not be recovered. This is due to the fact that the pressure in Equation (3.a.29) is the thermodynamic pressure while Hooke's law does not attempt to include thermodynamic effects. If nonadiabatic terms are included so as to produce a modified Hooke's law then the relation (3.a.21) would be modified to include temperature effects while the stress deviators S_{ij} would be unchanged. Hence the quantities S_{ij} in Equations (3.a.22)-(3.a.29) are the stress deviators and an additional constant, i.e. k in Equation (3.a.20) (in addition to the shear modulus μ), usually encountered in two dimensional linear elasticity, is eliminated by the inclusion of an equation of state (3.a.29).

b) A Plastic Model

Equations (3.a.26)-(3.a.28) express the stress-strain relations for a material behaving with linear elastic properties. Before we proceed, we first rewrite these equations in a more convenient form. We define the derivative $\frac{D}{Dt}$ to be a tensor operator unaffected by rotations. Then, we construct this operator from the substantial derivative of the stress deviators

$$
\begin{aligned}
\frac{DS_{11}}{Dt} &= \frac{dS_{11}}{dt} - S_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \\
\frac{DS_{12}}{Dt} &= \frac{dS_{12}}{dt} + \frac{S_{11}-S_{22}}{2}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right) \qquad (3.b.1) \\
\frac{DS_{22}}{Dt} &= \frac{dS_{22}}{dt} + S_{12}\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)
\end{aligned}
$$

The time derivatives of the strain deviators can be written using the definitions:

$$
\begin{aligned}
\dot{\varepsilon}_{11} &= \frac{\partial u}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) \\
&= \frac{2}{3}\frac{\partial u}{\partial z} - \frac{1}{3}\frac{\partial v}{\partial r} - \frac{1}{3}\frac{v}{r} \qquad (3.b.2) \\
\dot{\varepsilon}_{22} &= \frac{\partial v}{\partial r} - \frac{1}{3}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) \\
&= -\frac{1}{3}\frac{\partial u}{\partial z} + \frac{2}{3}\frac{\partial v}{\partial r} - \frac{1}{3}\frac{v}{r} \\
\dot{\varepsilon}_{12} &= \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)
\end{aligned}
$$

Then, Equations (3.a.26)-(3.a.28) can be written so that the deviator stress components of the stress tensor are obtained from the deviator strain components of the strain tensor, i.e.,

$$\frac{DS_{11}}{Dt} = 2\mu\dot{\varepsilon}_{11}$$

$$\frac{DS_{12}}{Dt} = 2\mu\dot{\varepsilon}_{12} \qquad (3.b.3)$$

$$\frac{DS_{22}}{Dt} = 2\mu\dot{\varepsilon}_{22}$$

Equations (3.b.3) are applicable only in the elastic region of flow. In general a material which exhibits a linear variation of strain with stress is called linear elastic. However at the proportional limit the strain may increase more rapidly with increasing stress. In this region, the material deforms plastically. If the strain is allowed to increase with no increase in the stress the material is called perfectly plastic. If some variation in stress occurs the material is experiencing work hardening.

A material when exposed to external loading can experience permanent deformation as stresses exceed certain characteristic limits of the material. A tacit assumption is made in elastic theory: the assumption that a scalar function $f(\tau_{ij}, \varepsilon^p_{ij}, \omega)$, called a yield function, exists. Arguments τ_{ij}, ε^p_{ij} and ω correspond to the stress state, the plastic strain and a measure of the loading history respectively.

The equation

$$f = 0$$

represents a surface in stress space; for $f < 0$ the change in plastic deformation is zero while only when $f=0$ is plastic deformation allowed to occur. If the material properties are independent of strain rate, $f>0$ has no meaning. In the plastic region, in place of System (3.b.3), we invoke the Prandtl-Reuss[2] formulation for plastic flow.

In a mixed elastic plastic flow material, System (3.b.3) applies whenever

$$\sum_{1 \leq i,\, j \leq 3} S^2_{ij} = 2\,(S^2_{11} + S^2_{22} + S_{11}\,S_{22} + S^2_{12}) < 2K^2 \qquad (3.b.4)$$

with K^2 a constant of the material. However, whenever the von Mises yield condition, based on the assumed form for the yield function $f=f(\Sigma S_{ij})$, requires that

$$\Sigma S^2_{ij} \geq 2K^2 \qquad (3.b.5)$$

be satisfied, then System (3.b.3) is replaced by a viscoelastic model system patented after a viscoelastic constitutive relation of the form

$$\frac{DS_{11}}{Dt} = 2\mu\,(\dot{\varepsilon}_{11} - \lambda S_{11})$$

$$\frac{DS_{12}}{Dt} = 2\mu\,(\dot{\varepsilon}_{12} - \lambda S_{12}) \qquad (3.b.6)$$

$$\frac{DS_{22}}{Dt} = 2\mu\,(\dot{\varepsilon}_{22} - \lambda S_{22})$$

Now the constant λ is determined by requiring equality in the von Mises yield criterion (3.b.5) rather than setting it to 1/2 $\rho\nu$, ν the kinematic viscosity of the material. Multiply each Equation (3.b.6) by S_{ij} and sum:

$$\frac{1}{2}\ \Sigma\frac{DS_{ij}^2}{Dt} = 2\mu\ \ (\Sigma S_{ij}\dot{\varepsilon}_{ij} - \lambda\Sigma S_{ij}^2) \qquad (3.b.7)$$

Now use the fact that $\Sigma S_{ij}^2 = 2K^2$ = constant. Equation (3.b.7) can then be solved explicitly for λ,

$$\lambda = \frac{1}{2K^2}\,\Sigma S_{ij}\dot{\varepsilon}_{ij} \qquad (3.b.8)$$

In cylindrical coordinates (3.b.8) can be expressed as

$$\lambda = \frac{1}{2K^2}\left\{S_{11}\left[u_z - 1/3\ (u_{zz} + v_{rr} + \frac{v}{r})\right]\right.$$

$$+ S_{22}\left[v_r - 1/3\ (u_{zz} + v_{rr} + \frac{v}{r})\right]$$

$$\left. + S_{12}\ (u_r + v_z) - (S_{11} + S_{22})\left[\frac{v}{r} - 1/3\ (u_{zz} + v_{rr} + \frac{v}{r})\right]\right\}$$

$$= \frac{1}{2K^2}\left\{S_{11}\,u_{zz} + S_{22}\,v_{rr} + S_{12}\,(\frac{u_r+v_z}{2})\right.$$

$$\left. - (S_{11} + S_{22})\,\frac{v}{r}\right\} \qquad (3.b.8')$$

Here we have used the notation $\frac{\partial u}{\partial z} = u_z$, etc.

In this way both the elastic and plastic regions can be described by Equation (3.b.6). The prescription is

$$\text{Elastic}\begin{cases} \Sigma S_{ij}^2 < 2K^2 & \text{or} \quad \Sigma S_{ij}^2 = 2K^2 \text{ and} \\ \lambda = 0 & \quad \Sigma S_{ij}\dot{\varepsilon}_{ij} \leq 0 \text{ (unloading)} \end{cases} \tag{3.b.9a}$$

$$\text{Plastic}\begin{cases} \Sigma S_{ij}^2 = 2K^2 \\ \lambda = \dfrac{1}{2K^2}\,\Sigma S_{ij}\dot{\varepsilon}_{ij} > 0 \end{cases} \tag{3.b.9b}$$

Equations (3.b.9a) and (3.b.9b) show that, in the plastic region, if one begins on the material yield surface, and in the absence of unloading, then one remains on the yield surface.

Because of the complicated boundary conditions together with the nonlinearities of Equations (3.b.9a) and (3.b.9b) this system must be solved numerically. If one uses a finite difference technique in the plastic region then the truncation errors inherent in any difference scheme will result in a set of deviatoric stresses which no longer lie on the yield surface. It is therefore necessary to change the finite difference schemes in the plastic region to insure that unloading does not occur due to truncation errors. Thus, λ in the numerical method will not strictly be determined by Equation (3.b.9b) but instead the derivation of this formula will be used to force the yield condition to be satisfied numerically.

In order to describe the method used, which is second order accurate, we assume that a solution is known at time t and we wish to determine the solution at time t+Δt. The solution to Equation (3.b.10) with λ=0 (i.e. the elastic case) will be denoted by S_{ij}^e. Then, by using a backward Taylor Series in time one has

$$S_{ij}(t) = S_{ij}(t+\Delta t) - \Delta t\, \dot{S}_{ij}(t+\Delta t) + \frac{(\Delta t)^2}{2}\, \ddot{S}_{ij}(t+\Delta t) + \mathcal{O}((\Delta t)^3)$$

or

$$S_{ij}(t+\Delta t) = S_{ij}(t) + \Delta t\, \dot{S}_{ij}(t+\Delta t) - \frac{(\Delta t)^2}{2}\, \ddot{S}_{ij}(t+\Delta t) + \mathcal{O}((\Delta t)^3) \tag{3.b.10}$$

Using the differential Equation (3.b.6) in Equation (3.b.10) yields

$$S_{ij}(t+\Delta t) = S_{ij}(t) + 2\Delta t\mu \left[\dot{\varepsilon}_{ij}(t+\Delta t) - \lambda(t+\Delta t)S_{ij}(t+\Delta t)\right] - 2\mu\frac{(\Delta t)^2}{2}\left[\dot{\varepsilon}_{ij}(t+\Delta t) - \frac{\lambda(t+\Delta t)S_{ij}(t+\Delta t)-\lambda(t)S_{ij}(t)}{\Delta t}\right] \quad (3.b.11)$$

Or introducing the elastic deviatoric stresses, S^e_{ij}, Equation (3.b.11) can be written as

$$S_{ij}(t+\Delta t) = S^e_{ij}(t+\Delta t) - 2\Delta t\mu\,\lambda(t+\Delta t)\,S_{ij}(t+\Delta t) + \Delta t\mu\left[\lambda(t+\Delta t)S_{ij}(t+\Delta t) - \lambda(t)S_{ij}(t)\right] \quad (3.b.12)$$

Because all terms containing $S_{ij}(t+\Delta t)$ are linear, we may solve directly for the predicted deviatoric stress at the advanced time level via

$$S_{ij}(t+\Delta t) = \alpha\left[S^e_{ij}(t+\Delta t) - \Delta t\mu\lambda(t)S_{ij}(t)\right] \quad (3.b.13)$$

All terms on the right hand side of Equation (3.b.13) are known except for α. We determine α by requiring the $S_{ij}(t+\Delta t)$ to lie on the yield surface. As before we square Equation (3.b.13) and sum over i and j. Then

$$2K^2 = \sum_{i,j} S^2_{i,j}(t+\Delta t) = \alpha^2 \sum_{i,j}\left[S^e_{i,j}(t+\Delta t) - (\Delta t)\mu\lambda(t)S_{ij}(t)\right]^2$$

Solving for α we obtain

$$\alpha = \frac{K}{\sqrt{1/2\sum_{i,j}\left[S^e_{ij}(t+\Delta t)-\Delta t\mu\lambda(t)S_{ij}(t)\right]^2}} \quad (3.b.14)$$

To sum up, the procedure for solving Equation (3.b.6) is given by the following three step algorithm:

i) Determine $S^e_{ij}(t+\Delta t)$ by solving Equation (3.b.6), by any second order method, with $\lambda=0$.

ii) Test if $\Sigma \left[S^e_{ij}(t+\Delta t)\right]^2 < 2K^2$. If true, set $S_{ij}(t+\Delta t) = S^e_{ij}(t+\Delta t)$ otherwise determine α from Equation (3.b.14).

iii) Finally solve for the deviatoric stresses at the advanced time level using

$$S_{ij}(t+\Delta t) = \alpha \left[S^e_{ij}(t+\Delta t) - (\Delta t)\mu\lambda(t)S_{ij}(t)\right] \qquad (3.b.15)$$

with

$$\lambda = (2K^2)^{-1} \sum_{\substack{1 \le m \le 3 \\ 1 \le n \le 3}} S_{mn}(t)\dot{\varepsilon}_{mn}(t)$$

c) <u>Transformed Differential Equation</u>

The partial differential equations described in section (3a), Equations (3.a.22)-(3.a.28), can be written in quasilinear form, i.e,

$$w_t + A\,w_z + B\,w_r + \frac{1}{r}\,Cw = 0 \qquad (3.c.1)$$

A, B and C are 7x7 matricies; their entries are given below:

$$A=\begin{pmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & -1/\rho & 0 & 0 \\ 0 & 0 & u & 0 & 0 & -1/\rho & 0 \\ 0 & -\tau_{11}/\rho & -\tau_{12} & u & 0 & 0 & 0 \\ 0 & -\frac{4}{3}\mu & \tau_{12} & 0 & u & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\tau_{11}-\tau_{22}) & 0 & 0 & u & 0 \\ 0 & \frac{2}{3}\mu & -\tau_{12} & 0 & 0 & 0 & u \end{pmatrix}$$

$$B=\begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -1/\rho & 0 \\ 0 & 0 & v & 0 & 0 & 0 & -\tau_{22}/\rho \\ 0 & -\tau_{12} & -\tau_{22}/\rho & v & 0 & 0 & 0 \\ 0 & -\tau_{12} & 2\mu/3 & 0 & v & 0 & 0 \\ 0 & \frac{1}{2}(\tau_{11}-\tau_{22})-\mu & 0 & 0 & 0 & v & 0 \\ 0 & \tau_{12} & \frac{-4\mu}{3} & 0 & 0 & 0 & v \end{pmatrix}$$

where $\tau_{ij} = S_{ij} - p\delta_{ij}$ and finally

$$C=\begin{pmatrix} v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\ 0 & 0 & 0 & 0 & -1/\rho & 0 & -2/\rho \\ 0 & S_{11}+S_{22}+p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\mu}{3}v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\mu}{3}v & 0 & 0 & 0 & 0 \end{pmatrix}$$

In Equation (3.c.1) w is the seven vector

$$w=\begin{pmatrix} \rho \\ u \\ v \\ e \\ S_{11} \\ S_{12} \\ S_{22} \end{pmatrix} \qquad (3.c.2)$$

If one introduces a general transformation

$$\alpha = \alpha(z,r) \qquad (3.c.3)$$
$$\beta = \beta(z,r)$$

we can, by the chain rule rewrite (3.c.1) in the α-β plane:

$$w_t + (A\alpha_z + B\alpha_r)\, w_\alpha + (B\beta_r + A\beta_z)\, w_\beta$$
$$+ \frac{1}{r(\alpha,\beta)} Cw=0. \qquad (3.c.4)$$

In order to solve the system (3.c.4) by the second order Lax-Wendroff[3] method, which uses a Taylor expansion for the solution vector $w(t+\Delta t)$ about the initial data $w(t)$ via

$$w(t+\Delta t) = w(t) + \Delta t\, w_t + \frac{\Delta t^2}{2} w_{tt} + \mathcal{O}(\Delta t^3) \qquad (3.c.5)$$

It is necessary to compute the second time derivative w_{tt}; the first time derivative is obtained directly from (3.c.4).

It is sufficient to show how this is accomplished for a single component of w. For example the equation for the density, from Equation (3.c.1), is just

$$\rho_t + u\rho_z + v\rho_r + \rho u_z + \rho v_r + \frac{\rho v}{r} = 0 \qquad (3.c.6)$$

The counterpart of Equation (3.c.6), in the (α-β) plane is just

$$\rho_t + \rho(\alpha_z u_\alpha + \alpha_r v_\alpha + \beta_r v_\beta + \beta_z u_\beta) + (u\alpha_z + v\alpha_r)\rho_\alpha$$

$$+ (v\beta_r + u\beta_z)\rho_\beta + \frac{\rho v}{r(\alpha,\beta)} = 0 \qquad (3.c.7)$$

Under the assumption that the coordinate system is independent of time we may compute the second time derivative of the density from Equation (3.c.7):

$$\begin{aligned}\rho_{tt} = &- \rho_t(\alpha_z u_\alpha + \alpha_r v_\alpha + \beta_r v_\beta + \beta_z u_\beta)\\ &- \rho(\alpha_z u_{t\alpha} + \alpha_r v_{t\alpha} + \beta_r v_{t\beta} + \beta_z u_{t\beta})\\ &- (u_t\alpha_z + v_t\alpha_r)\rho_\alpha - \rho_{t\alpha}(u\alpha_z + v\alpha_r) \qquad (3.c.8)\\ &- (v_t\beta_r + u_t\beta_z)\rho_\beta - (v\beta_r + u\beta_z)\rho_{t\beta}\\ &- \frac{1}{r(\alpha,\beta)}(\rho_t v + v\rho_t) = 0\end{aligned}$$

Now the terms on the right hand side of Equation (3.c.8) can be obtained from Equation (3.c.4) by taking the appropriate space differential. For example the terms involving $\rho_{t\alpha}$ (and $\rho_{t\beta}$) are evaluated by operating on the first (second) component of System

(3.c.4) with the differential operators $\frac{\partial}{\partial\alpha}$ $(\frac{\partial}{\partial\beta})$. Cross derivatives of the velocity components are evaluated by taking appropriate space derivatives of the second and third components of the System (3.c.4). In this way we find that only spatial derivatives appear in the differential relation describing the second time derivative of the density,

$$\rho_{tt} = P(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial^2}{\partial\alpha^2}, \frac{\partial^2}{\partial\alpha\partial\beta}, \frac{\partial^2}{\partial\beta})w \tag{3.c.9}$$

Here P is a second order differential operator acting on w. In this manner all components of the vector w_{tt} may be computed.

In the present study we considered the class of transformations to be restricted under the assumptions

$$\begin{aligned} \alpha_z &= 1 \\ \beta_z &= 0 \end{aligned} \tag{3.c.10}$$

Thus β is associated only with r while α can depend on both r and z. Equation (3.c.7) is then simplified:

$$\begin{aligned} \rho_t = &- (u + \alpha_r v)\rho_\alpha - \rho(u_\alpha + \alpha_r v_\alpha) \\ &- \beta_r(\rho v_\beta + v\rho_\beta) - \frac{\rho v}{r(\alpha,\beta)} \end{aligned} \tag{3.c.11}$$

We now write down the remaining components of System (3.c.4) under condition (3.c.10):

The axial momentum equation is used to obtain the rate of change of the axial velocity through

$$\begin{aligned} u_t = &- (u + \alpha_r v)u_\alpha + \frac{1}{\rho}(\tau_{11,\alpha} + \alpha_r S_{12,\alpha}) \\ &- \beta_r (vu_\beta - \frac{S_{12,\beta}}{\rho}) + \frac{S_{12}}{\rho r} \end{aligned} \tag{3.c.12}$$

The radial momentum equation is used to obtain the rate of change of the radial velocity via

$$\begin{aligned} v_t = &- (u + \alpha_r v)v_\alpha + \frac{1}{\rho}(S_{12,\alpha} + \alpha_r \tau_{22,\alpha}) \\ &- \beta_r(v - v_\beta - \frac{\tau_{22,\beta}}{\rho}) + \frac{2S_{22}+S_{11}}{\rho r} \end{aligned} \tag{3.c.13}$$

The rate of change of the specific internal energy is determined from

$$e_t = -(u + \alpha_r v)e_\alpha + \frac{1}{\rho}\Big[(\tau_{11} + \alpha_r S_{12})u_\alpha + (S_{12} + \alpha_r \tau_{22})v_\alpha\Big] - \beta_r\Big[v e_\beta - \frac{1}{\rho}(\tau_{22}v_\beta + S_{12}u_\beta)\Big] - \frac{v(p+S_{11}+S_{22})}{\rho r} \quad (3.c.14)$$

The final three equations (3.c.15), (3.c.16) and (3.c.17) determine the rate of change of the deviatoric stresses S_{11}, S_{12} and S_{22} respectively. The notation S_{11} corresponds to S_{zz}, S_{12} to S_{zr} and S_{22} to S_{rr}:

$$S_{11,t} = -(u + \alpha_r v)S_{11,\alpha} + (\frac{4\mu}{3} + \alpha_r S_{12})u_\alpha - (S_{12} + \alpha_r \frac{2\mu}{3})v_\alpha - \beta_r\Big[vS_{11,\beta} - S_{12}u_\beta + \frac{2\mu}{3} v_\beta\Big] - \frac{2\mu}{3}\frac{v}{r} \quad (3.c.15)$$

$$S_{12,t} = -(u + \alpha_r v)S_{12,\alpha} + \alpha_r (\mu - \frac{S_{11}-S_{22}}{2})u_\alpha + (\frac{S_{11}-S_{22}}{2} + \mu)v_\alpha - \beta_r\Big(vS_{12,\beta} + (\frac{S_{11}-S_{22}}{2} - \mu)u_\beta\Big) \quad (3.c.16)$$

$$S_{22,t} = -(u + \alpha_r v)S_{22,\alpha} - (\frac{2\mu}{3} + \alpha_r S_{12})u_\alpha + (S_{12} + \alpha_r \frac{4\mu}{3})v_\alpha - \beta_r\Big(vS_{22\beta} + S_{12}u_\beta - \frac{4\mu}{3}v_\beta\Big) - \frac{2\mu}{3}\frac{v}{r} \quad (3.c.17)$$

Equation (3.c.12) through (3.c.17) are valid for $\beta \neq 0$. For the special case of flow on the axis of symmetry, $\beta = 0$, L'hospitals rule when applied to the above system, Equations (3.c.11)-(3.c.17), yields

$$\rho_t = -\rho u_\alpha - u\rho_\alpha - 2\beta_r \rho v_\beta \tag{3.c.11a}$$

$$u_t = -uu_\alpha + \frac{1}{\rho}\tau_{11,\alpha} + 2\beta_r \frac{S_{12,\beta}}{\rho} \tag{3.c.12a}$$

$$v_t = 0 \tag{3.c.13a}$$

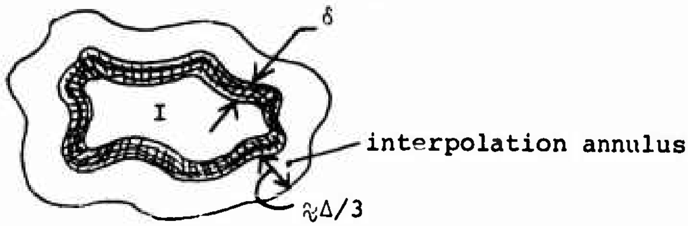
$$e_t = -ue_\alpha + \frac{1}{\rho}(\tau_{11}u_\alpha + 2\beta_r \tau_{22}v_\beta) \tag{3.c.14a}$$

$$S_{11,t} = -uS_{11,\alpha} + \frac{4\mu}{3}(u_\alpha - \beta_r v_\beta) \tag{3.c.15a}$$

$$S_{12,t} = 0 \tag{3.c.16a}$$

$$S_{22,t} = -uS_{22,\alpha} - \frac{2\mu}{3}(u_\alpha - \beta_r v_\beta) \tag{3.c.17a}$$

System (3.c.11) through (3.c.17) together with (3.c.11a) through (3.c.17a) are the differential equations solved in a strip of thickness $\delta \sim \sqrt{2}\Delta$, Δ the spatial step size near the boundary of the domain. This is shown in the figure below by the crosshatched area.



The thin annulus directly adjacent to the boundary is a region where mesh points lie too close to the boundary. Since the stability of the finite difference solution would not be satisfied in this region, interpolation between the boundary data and interior data is used to update the solution.

It now is appropriate to describe the form of the differential equations used for the solution interior to the domain (the region I in the above figure). Here we write the first four differential equations in conservation form choosing the entries of the vector w to be the quantities conserved across discontinuous transitions, i.e.,

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}$$

Here E is the sum of the specific internal and kinetic energy, the total energy, $E = \rho(e + 1/2(u^2 + v^2))$.

The continuity equation is

$$\rho_t + (\rho u)_z + (\rho v)_r + \frac{\rho v}{r} = 0 \qquad (3.c.18)$$

The axial momentum equation is

$$(\rho u)_t + (\rho u^2 - \tau_{11})_z + (\rho uv - S_{12})_r + \frac{(\rho uv - S_{12})}{r} = 0 \qquad (3.c.19)$$

The radial momentum equation is

$$(\rho v)_t + (\rho uv - S_{12})_z + (\rho v^2 - \tau_{22})_r + \frac{(\rho v^2 - 2S_{22} - S_{11})}{r} = 0 \qquad (3.c.20)$$

Conservation of energy requires that E satisfy

$$E_t + \Big[(E - \tau_{11})u - S_{12}v\Big]_z + \Big[(E - \tau_{22})v - S_{12}u\Big]_r + \frac{(E - \tau_{22})v - S_{12}u}{r} = 0 \qquad (3.c.21)$$

The stress strain relationships are not relationships that express a conservation principle. Hence they are rewritten here in their quasilinear component form:

$$S_{11,t} + uS_{11,z} + vS_{11,r} + \frac{2}{3}\mu(-2u_z + v_r + \frac{v}{r}) + S_{12}(v_z - u_r) = 0 \qquad (3.c.22)$$

$$S_{12,t} + uS_{12,z} + vS_{12,r} - \mu(u_r + v_z) + \frac{S_{11} - S_{22}}{2}(u_r - v_z) = 0 \qquad (3.c.23)$$

$$S_{22,t} + uS_{22,z} + vS_{22,r} + \frac{2\mu}{3}(-2v_r + u_z + \frac{v}{r}) + S_{12}(u_r - v_z) = 0 \qquad (3.c.24)$$

The specification of the equation of state, Equation (3.a.29), completes the system.

Along the axis of symmetry System (3.c.18) through (3.c.24) are redefined by the application of L'hospital's rule. Hence in z-r coordinate along the line r=0, we obtain the system

$$\rho_t + (\rho u)_z + 2(\rho v)_r = 0 \tag{3.c.18a}$$

$$(\rho u)_t + (\rho u^2 - \tau_{11})_z + 2(\rho uv - S_{12})_r = 0 \tag{3.c.19a}$$

$$(\rho v)_t = 0 \ , \ S_{11} + 2S_{22} = 0 \tag{3.c.20a}$$

$$E_t + \Big[(E - \tau_{11})u - S_{12}v\Big]_z + 2\Big[(E - \tau_{22})v - S_{12}u\Big]_r = 0 \tag{3.c.21a}$$

$$S_{11,t} + uS_{11,z} + \tfrac{4}{3}\mu(v_r - u_z) = 0 \tag{3.c.22a}$$

$$S_{12,t} = 0 \tag{3.c.23a}$$

$$S_{22,t} + uS_{22,z} + \tfrac{2}{3}\mu(u_z - v_r) = 0 \tag{3.c.24a}$$

In a situation where the transformation to the α-β plane is the identity transformation, the System (3.c.18) through (3.c.24) augmented on the line $\beta = r = 0$ with System (3.c.18a) through (3.c.24a) would be the complete set of interior equations to be solved. However for transformations used in the present work (Equations (3.c.3) subject to (3.c.10)) the conservation form for the equations in the α-β plane become

$$(r_\beta\rho)_t + \Big[r_\beta(\rho u + \alpha_r\rho v)\Big]_\alpha + \Big(\rho v\Big)_\beta + \frac{\rho v}{r\beta_r} = 0 \tag{3.c.25}$$

$$(r_\beta\rho u)_t + \Big[r_\beta(\rho u^2 - \tau_{11} + \alpha_r(\rho uv - S_{12}))\Big]_\alpha + \Big(\rho uv - S_{12}\Big)_\beta + \frac{\rho uv - S_{12}}{r\beta_r} = 0 \tag{3.c.26}$$

$$(r_\beta\rho v)_t + \Big[r_\beta(\rho uv - S_{12} + \alpha_r(\rho v^2 - \tau_{22}))\Big]_\alpha + \Big(\rho v^2 - \tau_{22}\Big)_\beta + \frac{\rho v^2 - 2S_{22} - S_{11}}{r\beta_r} = 0 \tag{3.c.27}$$

$$(r_\beta E)_t + \Big[r_\beta\ ((E - \tau_{11})u - S_{12}v + \alpha_r((E - \tau_{22})v - S_{12}u))\Big]_\alpha + \Big[(E - \tau_{22})v - S_{12}u\Big]_\beta + \frac{(E-\tau_{22})v - S_{12}u}{r\beta_r} = 0 \qquad (3.c.28)$$

The above four equations are augmented by the evolution equations for the deviatoric stresses in the α-β plane, Equations (3.c.15), (3.c.16) and (3.c.17).

Along the axis of symmetry, $\beta=0$, the above conservation laws reduce to

$$(r_\beta \rho)_t + \Big(r_\beta \rho u\Big)_\alpha + 2\Big(\rho v\Big)_\beta = 0 \qquad (3.c.25a)$$

$$(r_\beta \rho u)_t + \Big[r_\beta(\rho u^2 - \tau_{11})\Big]_\alpha + 2\Big(\rho uv - S_{12}\Big)_\beta = 0 \qquad (3.c.26a)$$

$$(r_\beta \rho v)_t = 0 \quad , \quad 2S_{22} + S_{11} = 0 \qquad (3.c.27a)$$

$$(r_\beta E)_t + \Big[r_\beta((E - \tau_{11})u)\Big]_\alpha + 2\Big[(E - \tau_{22})\ v - S_{12}u\Big]_\beta = 0 \qquad (3.c.28a)$$

The deviatoric stress components completing System (3.c.25a)-(3.c.28a) are determined by (3.c.15a), (3.c.16a) and (3.c.17a).

IV. Finite Difference Equations

The partial differential equations described in Section III have been written in both conservation form and quasilinear form in the computational (α-β) plane. This is necessary since one set, the conservation form, is used at all points which are interior to the domain and contain all their eight nearest neighbors interior to the domain. The quasilinear form is used to construct the difference scheme to be used at interior mesh points which have at least one of the nearest neighbors exterior to the domain. We start the discussion with the main difference scheme used, the two step method for the conservation form of the defining partial differential equations.

a) Two Step Method

We wish to solve the set of equations defined in the previous section, Equations (3.c.18)-(3.c.24), on a set of mesh points

$$\begin{aligned} \alpha_i &= i\Delta\alpha \ , \ i = 0,1,\ldots,I \\ \beta_j &= j\Delta\beta \ , \ j = 0,1,\ldots,J \qquad (4.a.1) \\ t_n &= n\Delta t \ , \ n = 0,1,\ldots \end{aligned}$$

For convenience we introduce the notation for the first four equations (3.c.18-3.c.21). Let f,g and h be four vectors defined by

$$f = \begin{pmatrix} \rho u \\ \rho u^2-\tau_{11} \\ \rho uv-S_{12} \\ (E-\tau_{11})u-S_{12}v \end{pmatrix}, \quad g = \begin{pmatrix} \rho v \\ \rho uv-S_{12} \\ \rho v^2-\tau_{22} \\ (E-\tau_{22})v-S_{12}u \end{pmatrix}$$

and

$$h = \begin{pmatrix} \rho v \\ \rho uv-S_{12} \\ \rho v^2-2S_{22}-S_{11} \\ (E-\tau_{22})v-S_{12}u \end{pmatrix} \qquad (4.a.2)$$

Then with $w^T = (\rho,\rho u,\rho v, E)$ we have

$$w_t + f_z + g_r + \frac{1}{r} h = 0 \qquad r \neq 0 \qquad (4.a.3)$$

$$w_t + f_z + (g+h)_r = 0 \qquad r = 0 \qquad (4.a.3a)$$

In the α-β plane (4.a.3) becomes, after application of the chain rule

$$w_t + f_\alpha + \alpha_r g_\alpha + \beta_r g_\beta + \frac{h}{r(\beta)} = 0$$

or

$$\left(\frac{w}{\beta_r}\right)_t + \left(\frac{\tilde{f}}{\beta_r}\right)_\alpha + g_\beta + \frac{h}{r\beta_r} = 0 \qquad (4.a.4)$$

where $\tilde{f} = f + \alpha_r g$.

Comparison of Equation (4.a.4) with the component forms, Equations (3.c.25)-(3.c.28), give the individual entries for the new flux vector $\tilde{f}$ and g. For the remainder of the discussion we drop the tilda on f.

The approximate solution is called V;

$$V(\alpha_i,\beta_j,t_n) = V^n_{ij} \approx w(\alpha_i,\beta_j,t_n) \qquad (4.a.5)$$

The approximate solution is written as a two step difference equation. Predicted values $\overline{V}$ are first obtained at the midpoints $(\alpha_{i+1/2},\beta_{j+1/2},t+n\Delta t)$ of the mesh by a first order difference approximation. These values are then used to obtain a second order accurate solution at regular mesh points. Letting $\lambda_1 = \Delta t/\Delta\alpha$ and $\lambda_2 = \Delta t/\Delta\beta$ the finite difference equation for the first step is

$$\bar{V}^{n}_{i+1/2,j+1/2} = 1/4(V^{n-1}_{i+1,j+1} + V^{n-1}_{i+1,j} + V^{n-1}_{i,j+1} + V^{n-1}_{i,j})$$

$$- 1/2\lambda_1(f^{n-1}_{i+1,j+1} - f^{n-1}_{i,j+1} + f^{n-1}_{i+1,j} - f^{n-1}_{i,j})$$

$$\qquad (4.a.6a)$$

$$- 1/2\lambda_2(g^{n-1}_{i+1,j+1} - g^{n-1}_{i+1,j} + g^{n-1}_{i,j+1} - g^{n-1}_{i,j})$$

$$- 1/4\Delta t(h^{n-1}_{i+1,j+1} + h^{n-1}_{i+1,j} + h^{n-1}_{i,j+1}$$

$$+ h^{n-1}_{i,j})/r(\beta_{j+1/2})$$

Introducing the notation $\bar{f}=f(\bar{V})$ the second step is defined by the finite difference equations

$$V^{n}_{i,j} = V^{n-1}_{i,j} - 1/4\lambda_1(f^{n-1}_{i+1,j} - f^{n-1}_{i-1,j} + \bar{f}^{n}_{i+1/2,j+1/2}$$

$$- \bar{f}^{n}_{i-1/2,j+1/2} + \bar{f}^{n}_{i+1/2,j-1/2} - \bar{f}^{n}_{i-1/2,j-1/2})$$

$$- 1/4\lambda_2(g^{n-1}_{i,j+1} - g^{n-1}_{i,j-1} + \bar{g}^{n}_{i+1/2,j+1/2}$$

$$- \bar{g}^{n}_{i+1/2,j-1/2} + \bar{g}^{n}_{i-1/2,j+1/2} - \bar{g}^{n}_{i-1/2,j-1/2}) \qquad (4.a.6b)$$

$$- 1/4\Delta t(h^{n-1}_{i,j+1} + h^{n-1}_{i,j-1} + 1/2(\bar{h}^{n}_{i+1/2,j+1/2}$$

$$+ \bar{h}^{n}_{i+1/2,j} + \bar{h}^{n}_{i-1/2,j+1/2} + \bar{h}^{n}_{i-1/2,j-1/2}))/r(\beta_j)$$

Stability of the above difference scheme is achieved if an artificial viscosity Q is added to the right hand side of Equation (4.a.6b)

$$Q = K \Big\{ \lambda_1 \Big[|\tilde{u}_{i+1,j} - \tilde{u}_{i,j}| \; (V_{i+1,j} - V_{i,j}) - |\tilde{u}_i - \tilde{u}_{i-1}| \; (V_{i,j} - V_{i-1,j}) \Big] + \lambda_2 \Big[|\tilde{v}_{i,j+1} - \tilde{v}_{i,j}| \, (V_{i,j+1} - V_{i,j}) - |\tilde{v}_{i,j} - \tilde{v}_{i,j-1}| \; (V_{i,j} - V_{i,j-1}) \Big] \Big\} \qquad (4.a.7)$$

where $\tilde{u}$ and $\tilde{v}$ are equal to $\dot{\alpha}$ and $\dot{\beta}$ respectively (Equation (5.a.1'). The time step Δt is kept at approximately 2/3 of the maximum allowable CFL value, i.e., $\Delta t = .65\ \Delta t_{CFL}$. We compute the Courant-Friedrichs-Lewey time step by finding η,

$$\eta = \max_{i,j} \left\{ (\tilde{u} + \sqrt{\alpha_r^2 + 1}\ C)/\Delta z,\ (\tilde{v} + \beta_r C)/\Delta r \right\}$$

over all mesh points. The maximum time step is then $\Delta t_{CFL} = 1/D\eta$.

Equations (3.c.22), (3.c.23) and (3.c.24) for the deviatoric stress components are solved in an entirely analogous manner. Now let $\bar{w}^T = (S_{11}, S_{12}, S_{22})$, $s^T = (\rho, u, v, e, S_{11}, S_{12}, S_{22})$, and

$$f = \begin{pmatrix} -\frac{4}{3}\mu u \\ -\mu v \\ \frac{2}{3}\mu u \end{pmatrix} ; \quad g = \begin{pmatrix} \frac{2}{3}\mu v \\ -\mu u \\ -\frac{4}{3}\mu v \end{pmatrix} ; \quad h = \begin{pmatrix} \frac{2}{3}\mu v \\ 0 \\ \frac{2}{3}\mu v \end{pmatrix} \qquad (4.a.8)$$

and define the matricies a and b as

$$a = \begin{pmatrix} 0 & 0 & S_{12} & 0 & u & 0 & 0 \\ 0 & 0 & -\dfrac{S_{11}-S_{22}}{2} & 0 & 0 & u & 0 \\ 0 & 0 & -S_{12} & 0 & 0 & 0 & u \end{pmatrix} \qquad (4.a.9)$$

and

$$
b = \begin{pmatrix} 0 & -S_{12} & 0 & 0 & v & 0 & 0 \\ 0 & \dfrac{S_{11}-S_{22}}{2} & 0 & 0 & 0 & v & 0 \\ 0 & S_{12} & 0 & 0 & 0 & 0 & v \end{pmatrix}
$$

Then introduce two transform matrices. A and B given in terms of a and b, $A = a + (a+\alpha_r b)/\beta_r$ and $B=b$. In vector notation the form used to generate the difference scheme, in terms of the above matrices, with $\tilde{f} = (f + \alpha_r g)/\beta_r$ is just

$$
\left(\frac{\bar{w}}{\beta_r}\right)_t + f_\alpha + g_\beta + Aw_\alpha + Bw_\beta + \frac{h}{r\beta_r} = 0 \qquad (4.a.10)
$$

If the terms with coefficients in Equation (4.a.10) when put into difference form are centered, the same two step algorithm (4.a.6a) and (4.a.6b) results for the stress deviators $\bar{w}$.

b) <u>One Step Method</u>

The basis for the one step algorithm is the method proposed by Lax and Wendroff[2]. As stated in Section 3, a Taylor series is used to determine the solution at time $t+\Delta t$ from known initial data at time t via

$$
w(t+\Delta t) = w(t) + \Delta t\, w_t + \frac{\Delta t^2}{2} w_{tt} + \mathcal{O}(\Delta t^3) \qquad (4.b.1)
$$

In the last section we described how the system of equations (3.c.11)-(3.c.17) is solved using (4.b.1). Reiterating briefly - one may solve the system (4.b.1) because w_t is determined directly from the differential equations. In order to evaluate the second time derivative one differentiates these same equations with respect to time. Two new cross derivatives appear - namely $w_{\alpha t}$ and $w_{\beta t}$; these latter two are determined by directly differentiating the same system (3.c.11)-(3.c.17) first with respect to α and then differentiating with respect to β. Back substitution of these two cross derivatives into the w_{tt} equations leads to a differential form for (4.b.1) the right hand side of which is completely independent of derivatives with respect to time.

We now plan to be quite specific. We will carry out all the indicated differentiation in the α-β plane for Equation (4.b.1).

The time derivative of the continuity equation (3.c.11) is

$$
\begin{aligned}
\rho_{tt} = &- (u + \alpha_r v)\rho_{\alpha t} - (u_t + \alpha_r v_t)\rho_\alpha - \rho_t(u_\alpha + \alpha_r v_\alpha) \\
&- \rho(u_{\alpha t} + \alpha_r v_{\alpha t}) - \beta_r \Big(\rho_t v_\beta + \rho v_{\beta t} + v\rho_{\beta t} \\
&+ v_t \rho_\beta\Big) - \frac{1}{r}(\rho_t v + \rho v_t)
\end{aligned} \qquad (4.b.2)
$$

Observe the appearance of αt and βt cross derivatives. They will be defined shortly.

The time derivative of the momentum equations (3.c.12) and (3.c.13) are

$$
\begin{aligned}
u_{tt} = &- (u + \alpha_r v)u_{\alpha t} - (u_t + \alpha_r v_t)u_\alpha \\
&+ \frac{1}{\rho}\left[\tau_{11,\alpha t} + \alpha_r S_{12,\alpha t} - \frac{\rho_t}{\rho}(\tau_{11,\alpha} + \alpha_r S_{12,\alpha})\right] \\
&- \beta_r \left\{v_t u_\beta + v u_{\beta t} - \frac{S_{12,\beta t}}{\rho} + \frac{\rho_t}{\rho^2} S_{12,\beta}\right\} \\
&+ \frac{1}{\rho r}(S_{12,t} - \frac{\rho_t}{\rho} S_{12})
\end{aligned} \qquad (4.b.3)
$$

and

$$
\begin{aligned}
v_{tt} = &- (u + \alpha_r v)v_{\alpha t} - (u_t + \alpha_r v_t)v_\alpha + \frac{1}{\rho}\Big[S_{12,\alpha t} \\
&+ \alpha_r \tau_{22,\alpha t} - \frac{\rho_t}{\rho}(S_{12,\alpha} + \alpha_r \tau_{22,\alpha})\Big] \\
&- \beta_r \left(v_t v_\beta + v v_{\beta t} - \frac{\tau_{22,\beta t}}{\rho} + \frac{\rho_t}{\rho^2}\tau_{22,\beta}\right) \\
&+ \frac{1}{\rho r}(2S_{22,t} + S_{11,t} - \frac{\rho_t}{\rho}(2S_{22} + S_{11}))
\end{aligned} \qquad (4.b.4)
$$

The cross derivatives in space and time also appear in the above two relations.

The second time derivative of the internal energy is the relative complicated expression

$$
\begin{aligned}
e_{tt} = &- (u + \alpha_r v)e_{\alpha t} - (u_t + \alpha_r v_t)e_\alpha \\
&+ \frac{1}{\rho}\Big[(\tau_{11,t} + \alpha_r S_{12,t})u_\alpha + (\tau_{11} + \alpha_r S_{12})u_{\alpha t} \\
&+ (S_{12,t} + \alpha_r \tau_{22,t})v_\alpha + (S_{12} + \alpha_r \tau_{22})v_{\alpha t} \\
&- \frac{\rho_t}{\rho}((\tau_{11} + \alpha_r S_{12})u_\alpha + (S_{12} + \alpha_r \tau_{22})v_\alpha)\Big] \\
&- \beta_r\Big[v_t e_\beta + v e_{\beta t} - \frac{1}{\rho}(\tau_{22,t}v_\beta + \tau_{22}v_{\beta t} \\
&+ S_{12,t}u_\beta + S_{12}u_{\beta t} - \frac{\rho_t}{\rho}(\tau_{22}v_\beta + S_{12}u_\beta))\Big] \\
&- \frac{1}{\rho r}\Big[v_t\,(p + S_{11} + S_{22} + v(p_t + S_{11,t} + S_{22,t}) \\
&- \frac{\rho_t}{\rho}\,v(p + S_{11} + S_{22})\Big]
\end{aligned}
\tag{4.b.5}
$$

The term p_t is evaluated by considering $\rho=\rho(t)$ and $e=e(t)$ in Equation (3.a.29); hence $p_t = P_\rho \rho_t + P_e e_t$.

The second time derivative of the deviatoric stress is

$$
\begin{aligned}
S_{11,tt} = &- (u + \alpha_r v)S_{11,\alpha t} - (u_t + \alpha_r v_t)S_{11,\alpha} \\
&+ \left(\frac{4\mu}{3} + \alpha_r S_{12}\right)u_{\alpha t} + \alpha_r S_{12,t}u_\alpha \\
&- \left(S_{12} + \alpha_r \frac{2\mu}{3}\right)v_{\alpha t} - S_{12,t}v_\alpha \\
&- \beta_r\Big(v_t S_{11,\beta} + v S_{11,\beta t} - S_{12,t}u_\beta \\
&- S_{12}u_{\beta t} + \frac{2\mu}{3}\,v_{\beta t}\Big) - \frac{2\mu}{3r}\,v_t
\end{aligned}
\tag{4.b.6}
$$

$$
\begin{aligned}
S_{12,tt} = &- (u + \alpha_r v) S_{12,\alpha t} - (u_t + \alpha_r v_t) S_{12,\alpha} \\
&+ \frac{\alpha_r}{2} (2\mu - S_{11} + S_{22}) u_{\alpha t} - \frac{\alpha_r}{2} (S_{11,t} \\
&- S_{22,t}) u_\alpha + 1/2\ (2\mu + S_{11} - S_{22}) v_{\alpha t} \\
&+ 1/2\ (S_{11,t} - S_{22,t}) v_\alpha - \beta_r \Big[v_t S_{12,\beta} \\
&+ v S_{12,\beta t} - 1/2\ (2\mu - S_{11} + S_{22}) u_{\beta t} \\
&+ 1/2\ (S_{11,t} - S_{22,t}) u_\beta \Big]
\end{aligned}
\tag{4.b.7}
$$

$$
\begin{aligned}
S_{22,tt} = &- (u + \alpha_r v) S_{22,\alpha t} - (u_t + \alpha_r v_t) S_{22,\alpha} \\
&- (\frac{2\mu}{3} + \alpha_r S_{12}) u_{\alpha t} - \alpha_r S_{12,t} u_\alpha + (S_{12} \\
&+ \alpha_r \frac{4\mu}{3}) v_{\alpha t} + S_{12,t} v_\alpha - \beta_r \Big[v_t S_{22,\beta} + v S_{22,\beta t} \\
&+ S_{12,t} u_\beta + S_{12} u_{\beta t} - \frac{4\mu}{3} v_{\beta t} \Big] - \frac{2\mu}{3} \frac{v_t}{r}
\end{aligned}
\tag{4.b.8}
$$

On the axis of symmetry, $\beta=0$, Equations (4.b.2)-(4.b.8) become

$$
\begin{aligned}
\rho_{tt} &= - u\rho_{\alpha t} - u_t \rho_\alpha - \rho_t u_\alpha - \rho u_{\alpha t} \\
&\quad - 2\beta_r (\rho_t v_\beta + \rho v_{\beta t})
\end{aligned}
\tag{4.b.2a}
$$

$$
\begin{aligned}
u_{tt} &= - u u_{\alpha t} - u_t u_\alpha + \frac{1}{\rho} (\tau_{11,\alpha t} - \frac{\rho_t}{\rho} \tau_{11,\alpha}) \\
&\quad + \frac{2\beta_r}{\rho} (S_{12,\beta t} - \frac{\rho_t}{\rho} S_{12,\beta})
\end{aligned}
\tag{4.b.3a}
$$

$$
v_{tt} = 0 \tag{4.b.4a}
$$

$$e_{tt} = - ue_{\alpha t} - u_t e_\alpha + \frac{1}{\rho} (\tau_{11,t} u_\alpha + \tau_{11} u_{\alpha t} - \frac{\rho_t}{\rho} \tau_{11} u_\alpha) + \frac{2\beta_r}{\rho} (\tau_{22,t} v_\beta + \tau_{22} v_{\beta t} - \frac{\rho_t}{\rho} \tau_{22} v_\beta) \quad (4.b.5a)$$

$$S_{11,tt} = - uS_{11,\alpha t} - u_t S_{11,\alpha} + \frac{4\mu}{3} (u_{\alpha t} - \beta_r v_{\beta t}) \quad (4.b.6a)$$

$$S_{12,tt} = 0 \quad (4.b.7a)$$

$$S_{22,tt} = - uS_{22,\alpha t} - u_t S_{22,\alpha} - \frac{2\mu}{3} (u_{\alpha t} - \beta_r v_{\beta t}) \quad (4.b.8a)$$

We must now compute the αt and βt cross derivatives which appear in the right hand side of the Systems (4.b.2)-(4.b.8) and (4.b.2a)-(4.b.8a); first the αt cross derivatives are computed followed by the βt derivatives.

The continuity equation yields for the cross derivative of the density $\rho_{\alpha t}$, for $\beta>0$,

$$\rho_{\alpha t} = - (u+\alpha_r v) \rho_{\alpha\alpha} - 2 (u_\alpha + \alpha_r v_\alpha) \rho_\alpha - \rho (u_{\alpha\alpha} + \alpha_r v_{\alpha\alpha}) - \beta_r \left[\rho_\alpha v_\beta + \rho v_{\alpha\beta} + v_\alpha \rho_\beta + v \rho_{\alpha\beta}\right] - \frac{\rho v_\alpha + \rho_\alpha v}{r} \quad (4.b.9)$$

while on the axis of symmetry, $\beta=0$, the density satisfies

$$\rho_{\alpha t} = - u\rho_{\alpha\alpha} - 2u_\alpha \rho_\alpha - \rho u_{\alpha\alpha} - 2\beta_r (\rho_\alpha v_\beta + \rho v_{\alpha\beta}) \quad (4.b.9a)$$

The cross derivative of the axial velocity $u_{\alpha t}$ for $\beta>0$, satisfies,

$$u_{\alpha t} = - (u + \alpha_r v) u_{\alpha\alpha} - (u_\alpha + \alpha_r v_\alpha) u_\alpha + \frac{1}{\rho} \left[\tau_{11,\alpha\alpha} + \alpha_r S_{12,\alpha\alpha} - \frac{\rho_\alpha}{\rho} (\tau_{11,\alpha} + \alpha_r S_{12,\alpha})\right] - \beta_r \left(v_\alpha u_\beta + v u_{\alpha\beta} - \frac{S_{12,\alpha\beta}}{\rho} + \frac{S_{12,\beta}\rho_\alpha}{\rho^2}\right) + \frac{1}{r} (\frac{S_{12,\alpha}}{\rho} - \frac{S_{12}}{\rho^2}\rho_\alpha) \quad (4.b.10)$$

while for $\beta=0$ (4.b.10) becomes

$$u_{\alpha t} = - uu_{\alpha\alpha} - u_\alpha u_\alpha + \frac{1}{\rho} (\tau_{11,\alpha\alpha} - \frac{\rho_\alpha}{\rho} \tau_{11,\alpha})$$

$$+ \frac{2\beta_r}{\rho} (S_{12,\alpha\beta} - \frac{\rho_\alpha}{\rho} S_{12,\beta}) \qquad (4.b.10a)$$

The cross derivative of the radial velocity $v_{\alpha t}$ for $\beta > 0$, satisfies,

$$v_{\alpha t} = - (u + \alpha_r v) v_{\alpha\alpha} - (u_\alpha + \alpha_r v_\alpha) v_\alpha$$

$$+ \frac{1}{\rho} \Big[S_{12,\alpha\alpha} + \alpha_r \tau_{22,\alpha\alpha} - \frac{\rho_\alpha}{\rho} (S_{12,\alpha}$$

$$+ \alpha_r \tau_{22,\alpha}) \Big] - \beta_r \Big(v_\alpha v_\beta + v v_{\alpha\beta} + \frac{\tau_{22,\alpha\beta}}{\rho} \qquad (4.b.11)$$

$$+ \frac{\tau_{22,\beta}\rho_\alpha}{\rho^2} \Big) + (\frac{2S_{22,\alpha} + S_{11,\alpha}}{\rho r}$$

$$- \frac{(2S_{22} + S_{11})\rho_\alpha}{r\rho^2})$$

while for $\beta=0$ (4.b.11) becomes

$$v_{\alpha t} = 0 \qquad (4.b.11a)$$

The cross derivative of the internal energy $e_{\alpha t}$ for $\beta > 0$, satisfies,

$$e_{\alpha t} = - (u + \alpha_r v) e_{\alpha\alpha} - (u_\alpha + \alpha_r v_\alpha) e_\alpha$$

$$+ \frac{1}{\rho} \Big[(\tau_{11,\alpha} + \alpha_r S_{12,\alpha}) u_\alpha + (\tau_{11} + \alpha_r S_{12}) u_{\alpha\alpha}$$

$$+ (S_{12,\alpha} + \alpha_r \tau_{22,\alpha}) v_\alpha + (S_{12} + \alpha_r \tau_{22}) v_{\alpha\alpha}$$

$$- \frac{\rho_\alpha}{\rho} ((\tau_{11} + \alpha_r S_{12}) u_\alpha + (S_{12} + \alpha_r \tau_{22}) v_\alpha) \Big] \qquad (4.b.12)$$

$$- \beta_r \Big[v_\alpha e_\beta + v e_{\alpha\beta} - \frac{1}{\rho} (\tau_{22,\alpha} v_\beta + \tau_{22} v_{\alpha\beta} + S_{12,\alpha} u_\beta$$

$$+ S_{12} u_{\alpha\beta}) - \frac{\rho_\alpha}{\rho} (\tau_{22} v_\beta + S_{12} u_\beta)) \Big] - \frac{1}{r} \Big[v_\alpha \frac{(p + S_{11} + S_{22})}{\rho}$$

$$+ v \frac{(p_\alpha + S_{11,\alpha} + S_{22,\alpha})}{\rho} - \frac{v\rho_\alpha (p + S_{11} + S_{22})}{\rho^2} \Big]$$

while for $\beta=0$ (4.b.12) becomes

$$e_{\alpha t} = - ue_{\alpha\alpha} - u_\alpha e_\alpha + \frac{1}{\rho} (\tau_{11,\alpha} u_\alpha + \tau_{11} u_{\alpha\alpha} - \frac{\rho_\alpha}{\rho} \tau_{11} u_\alpha) + \frac{2\beta_r}{\rho} (\tau_{22,\alpha} v_\beta + \tau_{22} v_{\alpha\beta} - \frac{\rho_\alpha}{\rho} \tau_{22} v_\beta) \quad (4.b.12a)$$

Again, the pressure derivative p_α encountered above is treated in precisely the same fashion as the derivative p_t described earlier. The cross derivative of the deviatoric stress $S_{11,\alpha t}$ for $\beta>0$, satisfies,

$$\begin{aligned} S_{11,\alpha t} = &- (u + \alpha_r v) S_{11,\alpha\alpha} - (u_\alpha + \alpha_r v_\alpha) S_{11,\alpha} \\ &+ (\frac{4\mu}{3} + \alpha_r S_{12}) u_{\alpha\alpha} + \alpha_r S_{12,\alpha} u_\alpha \\ &- (S_{12} + \alpha_r \frac{2\mu}{3}) v_{\alpha\alpha} - S_{12,\alpha} v_\alpha - \beta_r \Big(v_\alpha S_{11,\beta} \\ &+ v S_{11,\alpha\beta} - S_{12,\alpha} u_\beta - S_{12} u_{\alpha\beta} + \frac{2\mu}{3} v_{\alpha\beta} \Big) \\ &- \frac{2\mu}{3} \frac{v_\alpha}{r} \end{aligned} \quad (4.b.13)$$

while for $\beta=0$ (4.b.13) becomes

$$S_{11,\alpha t} = - uS_{11,\alpha\alpha} - u_\alpha S_{11,\alpha} + \frac{4\mu}{3} (u_{\alpha\alpha} - \beta_r v_{\alpha\beta}) \quad (4.b.13a)$$

The cross derivative of the deviatoric stress $S_{12,\alpha t}$ for $\beta>0$, satisfies,

$$\begin{aligned} S_{12,\alpha t} = &- (u + \alpha_r v) S_{12,\alpha\alpha} - (u_\alpha + \alpha_r v\ ) S_{12,\alpha} \\ &+ \frac{\alpha_r}{2} \Big[2\mu - (S_{11} - S_{22}) \Big] u_{\alpha\alpha} - \frac{\alpha_r}{2} (S_{11,\alpha} \\ &- S_{22,\alpha}) u_\alpha + 1/2 (S_{11} - S_{22} + 2\mu) v_{\alpha\alpha} \\ &+ 1/2 (S_{11,\alpha} - S_{22,\alpha}) v_\alpha - \beta_r \Big[v_\alpha S_{12,\beta} \\ &+ v S_{12,\alpha\beta} + 1/2 (S_{11} - S_{22} - 2\mu) u_{\alpha\beta} \\ &+ 1/2 (S_{11,\alpha} - S_{22,\alpha}) u_\beta \Big] \end{aligned} \quad (4.b.14)$$

while for $\beta=0$ (4.b.14) becomes

$$S_{12,\alpha t} = 0 \tag{4.b.14a}$$

The cross derivative of the deviatoric stress $S_{22,\alpha t}$ for $\beta>0$, satisfies,

$$\begin{aligned} S_{22,\alpha t} = &- (u + \alpha_r v)S_{22,\alpha\alpha} - (u_\alpha + \alpha_r v_\alpha)S_{22,\alpha} \\ &- (\tfrac{2\mu}{3} + \alpha_r S_{12})u_{\alpha\alpha} - \alpha_r S_{12,\alpha}u_\alpha + (S_{12} + \alpha_r \tfrac{4\mu}{3})v_{\alpha\alpha} \\ &+ S_{12,\alpha}v_\alpha - \beta_r \Big[v_\alpha S_{22,\beta} + vS_{22,\alpha\beta} + S_{12,\alpha}u_\beta \\ &+ S_{12}u_{\alpha\beta} - \frac{4\mu}{3} v_{\alpha\beta}\Big] - \frac{2\mu}{3}\frac{v_\alpha}{r} \end{aligned} \tag{4.b.15}$$

while for $\beta=0$ (4.b.15) becomes

$$S_{22,\alpha t} = - uS_{22,\alpha\alpha} - u_\alpha S_{22,\alpha} - \frac{2\mu}{3}\left(u_{\alpha\alpha} - \beta_r v_{\alpha\beta}\right) \tag{4.b.15a}$$

Now we state the results for differentiation of System (4.b.2)-(4.b.8) and (4.b.2a)-(4.b.8a) with respect to β.

The continuity equation yields for the cross derivative of the density $\rho_{\beta t}$, for $\beta>0$,

$$\begin{aligned} \rho_{\beta t} = &(u + \alpha_r v)\rho_{\alpha\beta} - (u_\beta + \alpha_r v_\beta)\rho_\alpha - \rho_\beta(u_\alpha + \alpha_r v_\alpha) \\ &- \rho(u_{\alpha\beta} + \alpha_r v_{\alpha\beta}) - \beta_r(2\rho_\beta v_\beta + \rho v_{\beta\beta} + v\rho_{\beta\beta}) \\ &- (\beta_r)_\beta(\rho v_\beta + v\rho_\beta) - \frac{1}{r}(\rho_\beta v + \rho v_\beta - \frac{r_\beta}{r}\rho v) \\ &- (\alpha_r)_\beta(v\rho_\alpha + \rho v_\alpha) \end{aligned} \tag{4.b.16}$$

while on the axis of symmetry, $\beta=0$, (4.b.16) reduces to

$$\rho_{\beta t} = - 2(\beta_r)_\beta \, \rho v_\beta \tag{4.b.16a}$$

Equation (4.b.16a) is not used since $(\beta_r)_\beta=0$.

The cross derivative of the axial velocity $u_{\beta t}$, for $\beta>0$, satisfies

$$
\begin{aligned}
u_{\beta t} = & - (u + \alpha_r v)u_{\alpha\beta} - (u_\beta + \alpha_r v_\beta)u_\alpha + \frac{1}{\rho}\Big[\tau_{11,\alpha\beta} \\
& + \alpha_r S_{12,\alpha\beta} - \frac{\rho_\beta}{\rho}(\tau_{11,\alpha} + \alpha_r S_{12,\alpha})\Big] \\
& - \beta_r\left(v_\beta u_\beta + vu_{\beta\beta} - \frac{S_{12,\beta\beta}}{\rho} + \frac{S_{12,\beta}\rho_\beta}{\rho^2}\right) \qquad (4.b.17) \\
& - (\beta_r)_\beta(vu_\beta - \frac{S_{12,\beta}}{\rho}) + \frac{1}{r}(\frac{S_{12,\beta}}{\rho} - \frac{S_{12,}\rho_\beta}{\rho^2} \\
& - \frac{S_{12}r_\beta}{\rho r}) - (\alpha_r)_\beta\left(vu_\alpha - \frac{S_{12,\alpha}}{\rho}\right)
\end{aligned}
$$

for $\beta>0$ while for $\beta=0$ (4.b.17) becomes

$$u_{\beta t} = 2(\beta_r)_\beta \frac{S_{12,\beta}}{\rho} \qquad (4.b.17a)$$

which is not used since u, as well as ρ, is an even function.

The cross derivative of the radial velocity $v_{\beta t}$, for $\beta>0$, satisfies

$$
\begin{aligned}
v_{\beta t} = & - (u + \alpha_r v)v_{\alpha\beta} - (u_\beta + \alpha_r v_\beta)v_\alpha \\
& + \frac{1}{\rho}\Big[S_{12,\alpha\beta} + \alpha_r\tau_{22,\alpha\beta} - \frac{\rho_\beta}{\rho}(S_{12,\alpha} + \alpha_r\tau_{22,\alpha})\Big] \\
& - \beta_r\left(v_\beta v_\beta + vv_{\beta\beta} - \frac{\tau_{22,\beta\beta}}{\rho} + \frac{\tau_{22,\beta}\rho_\beta}{\rho^2}\right) \qquad (4.b.18) \\
& - (\beta_r)_\beta(vv_\beta - \frac{\tau_{22,\beta}}{\rho}) + \frac{1}{r}\,\frac{2S_{22,\beta}+S_{11,\beta}}{\rho} \\
& - \frac{(2S_{22}+S_{11})\rho_\beta}{\rho^2} - \frac{(2S_{22}+S_{11})r_\beta}{\rho r} \\
& - (\alpha_r)_\beta(vv_\alpha - \frac{\tau_{22,\alpha}}{\rho})
\end{aligned}
$$

while for β=0 (4.b.18) becomes

$$v_{\beta t} = uv_{\alpha\beta} + \frac{1}{\rho}(S_{12,\alpha\beta} + (\alpha_r)_\beta \tau_{22,\alpha})$$

$$- \beta_r(v_\beta v_\beta - \frac{\tau_{22,\beta\beta} + 2S_{22,\beta\beta} + S_{11,\beta\beta}}{\rho}) \qquad (4.b.18a)$$

The cross derivative of the internal energy $e_{\beta t}$, $\beta>0$, satisfies the tedious relation

$$\begin{aligned}
e_{\beta t} = &- (u + \alpha_r v)e_{\alpha\beta} - (u_\beta + \alpha_r v_\beta)e_\alpha \\
&+ \frac{1}{\rho}\Big\{(\tau_{11,\beta} + \alpha_r S_{12,\beta})u_\alpha + (\tau_{11} + \alpha_r S_{12})u_{\alpha\beta} \\
&+ (S_{12,\beta} + \alpha_r \tau_{22,\beta})v_\alpha + (S_{12} + \alpha_r \tau_{22})v_{\alpha\beta} \\
&- \frac{\rho_\beta}{\rho}\Big[(\tau_{11} + \alpha_r S_{12})u_\alpha + (S_{12} + \alpha_r \tau_{22})v_\alpha\Big]\Big\} \\
&- \beta_r\Big[v_\beta e_\beta + ve_{\beta\beta} - \frac{1}{\rho}(\tau_{22,\beta}v_\beta + \tau_{22}v_{\beta\beta} \\
&+ S_{12,\beta}u_\beta + S_{12}u_{\beta\beta} - \frac{\rho_\beta}{\rho}(\tau_{22}v_\beta + S_{12}u_\beta))\Big] \\
&- (\beta_r)_\beta\Big[ve_\beta - \frac{1}{\rho}(\tau_{22}v_\beta + S_{12}u_\beta)\Big] \\
&- \frac{1}{\rho r}\Big[v_\beta\,(p + S_{11} + S_{22}) + v(p_\beta + S_{11,\beta} + S_{22,\beta}) \\
&- \frac{\rho_\beta}{\rho}v(p + S_{11} + S_{22}) - \frac{r_\beta}{r}\,v\,(p + S_{11} + S_{22})\Big] \\
&- (\alpha_r)_\beta\Big[ve_\alpha - \frac{1}{\rho}(S_{12}u_\alpha + \tau_{22}v_\alpha)\Big]
\end{aligned} \qquad (4.b.19)$$

which simplifies for β=0

$$e_{\beta t} = 2(\beta_r)_\beta \frac{\tau_{22}v_\beta}{\rho} \qquad (4.b.19a)$$

Again Equation (4.b.19a) is not used since e is an even function around β=0.

The cross derivative of the deviatoric stress $S_{11,\beta t}$, for $\beta>0$, satisfies

$$S_{11,\beta t} = -(u + \alpha_r v)S_{11,\alpha\beta} - (u_\beta + \alpha_r v_\beta)S_{11,\alpha}$$
$$+ (\tfrac{4\mu}{3} + \alpha_r S_{12})u_{\alpha\beta} + \alpha_r S_{12,\beta}u_\alpha$$
$$- (S_{12} + \alpha_r \tfrac{2\mu}{3})v_{\alpha\beta} - S_{12,\beta}v_\alpha - \beta_r \Big(v_\beta S_{11,\beta}$$
$$+ vS_{11,\beta\beta} - S_{12,\beta}u_{\beta\beta} - S_{12}u_{\beta\beta} + \tfrac{2\mu}{3} v_{\beta\beta}\Big) \qquad (4.b.20)$$
$$- (\beta_r)_\beta\Big(vS_{11,\beta} - S_{12}u_\beta + \tfrac{2\mu}{3} v_\beta\Big)$$
$$- \tfrac{2\mu}{3}\,\tfrac{1}{r}\,\Big(v_\beta - \tfrac{r_\beta v}{r}\Big) - (\alpha_r)_\beta \Big(vS_{11,\alpha}$$
$$- S_{12}u_\alpha + \tfrac{2\mu}{3} v_\alpha\Big)$$

while for $\beta=0$ (4.b.20) becomes

$$S_{11,\beta t} = -\tfrac{4\mu}{3}(\beta_r)_\beta\, v_\beta \qquad (4.b.20a)$$

Again (4.b.20a) is not used.

The cross derivative of the deviatoric stress $S_{12,\beta t}$, for $\beta>0$, satisfies

$$S_{12,\beta t} = -(u + \alpha_r v)S_{12,\alpha\beta} - (u_\beta + \alpha_r v_\beta)S_{12,\alpha}$$
$$+ \tfrac{\alpha_r}{2}(2\mu - S_{11} + S_{22})u_{\alpha\beta} - \tfrac{\alpha_r}{2}(S_{11,\beta} - S_{22,\beta})u_\alpha$$
$$+ 1/2\,(2\mu + S_{11} - S_{22})v_{\alpha\beta} + 1/2\,(S_{11,\beta} - S_{22,\beta})v_\alpha$$
$$- \beta_r\Big[v_\beta S_{12,\beta} + vS_{12,\beta\beta} - 1/2\,(2\mu - S_{11} + S_{22})u_{\beta\beta}$$
$$+ 1/2\,(S_{11,\beta} - S_{22,\beta})u_\beta\Big] - (\beta_r)_\beta\Big(vS_{12,\beta} \qquad (4.b.21)$$
$$- 1/2\,(2\mu - S_{11} + S_{22})u_\beta\Big) - (\alpha_r)_\beta\Big[vS_{12,\alpha}$$
$$- 1/2\,(2\mu - S_{11} + S_{22})u_\alpha\Big]$$

while for $\beta=0$ (4.b.21) becomes

$$S_{12,\beta t} = - uS_{12,\alpha\beta} + (\mu + \frac{S_{11}-S_{22}}{2})v_{\alpha\beta}$$
$$+ (\alpha_r)_\beta(\mu - \frac{S_{11}-S_{22}}{2})u_\alpha \qquad (4.b.21a)$$
$$- \beta_r \left[v_\beta S_{12,\beta} - (\mu - \frac{S_{11}-S_{22}}{2})u_{\beta\beta}\right]$$

Finally, the cross derivative of the deviatoric stress $S_{22,\beta t}$ for $\beta>0$, satisfies

$$S_{22,\beta t} = - (u + \alpha_r v)S_{22,\alpha\beta} - (u_\beta + \alpha_r v_\beta)S_{22,\alpha}$$
$$- (\frac{2\mu}{3} + \alpha_r S_{12})u_{\alpha\beta} - \alpha_r S_{12,\beta}u_\alpha + (S_{12}$$
$$+ \alpha_r \frac{4\mu}{3}) v_{\alpha\beta} + S_{12,\beta}v_\alpha - \beta_r \Big(v_\beta S_{22,\beta}$$
$$+ vS_{22,\beta\beta} + S_{12,\beta}u_\beta + S_{12}u_{\beta\beta} - \frac{4\mu}{3} v_{\beta\beta}\Big) \qquad (4.b.22)$$
$$- \frac{2\mu}{3}\frac{1}{r} (v_\beta - \frac{r_\beta}{r} v) - (\beta_r)_\beta \Big(vS_{22,\beta}$$
$$+ S_{12}u_\beta - \frac{4\mu}{3} v_\beta\Big) - (\alpha_r)_\beta \Big(vS_{22,\alpha}$$
$$+ S_{12}u_\alpha - \frac{4\mu}{3} v_\alpha\Big)$$

while for $\beta=0$ (4.b.22) becomes

$$S_{22,\beta t} = (\beta_r)_\beta \frac{2\mu}{3} v_\beta \qquad (4.b.22a)$$

Again (4.b.22a) is not used since S_{22} is an even function about $\beta=0$.

It is clear then that Equation (3.c.9) may now be generalized to read

$$w_{tt} = P(\partial_\alpha, \partial_\beta, \partial^2_{\alpha\alpha}, \partial^2_{\alpha\beta}, \partial^2_{\beta\beta})w \qquad (4.b.23)$$

Since w_{tt} determined from Equation (4.b.23) is multiplied by Δt^2 upon substitution into Equation (4.b.1) it is only necessary to evaluate the finite difference approximation of the spatial differential operator P to first order accuracy. It is precisely this fact that allows us to use uncentered finite difference approximations to spatial derivatives of w and maintain the second order accuracy of the overall scheme. Conversely it is precisely the lack of a fixed regular stencil caused by the boundary of the domain cutting the Eulerian mesh which leads to the relaxation of requiring and attaining a second order approximation to the second space derivatives.

Whenever the appropriate neighbors are available the spatial derivatives ∂_α, $\partial^2_{\alpha\alpha}$, ∂_β and $\partial^2_{\beta\beta}$ are approximated by centered differences which are second order accurate. For example, the component u_α is approximated by

$$\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta\alpha}$$

and u is approximated by

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta\alpha)^2}$$

and the error made through this approximation is of the order of $(\Delta\alpha)^3$.

Whenever the points (i+1,j) or (i-1,j) are missing, these formulas are replaced by noncentered formulas of second order accuracy. For example, if the point (i+1,j) is missing and the distance between the point (i,j) and the boundary (along an α coordinate) is $\alpha_1>0$ then u_α is approximated by

$$u_\alpha \sim \alpha\frac{\Delta\alpha}{\alpha_1(\alpha_1+\Delta\alpha)}\, u_B + \frac{\alpha_1-\Delta\alpha}{\alpha_1\Delta\alpha}\, u_{i,j} - \frac{\alpha}{\Delta\alpha(\alpha_1+\Delta\alpha)}\, u_{i-1,j} \qquad (4.b.24)$$

while $u_{\alpha\alpha}$ is approximated by

$$u_{\alpha\alpha} \sim 2\left\{\frac{u_B}{\alpha_1(\alpha_1+\Delta\alpha)} - \frac{u_{i,j}}{\alpha_1\Delta\alpha} + \frac{u_{i-1,j}}{\Delta\alpha(\alpha_1+\Delta\alpha)}\right\} \qquad (4.b.25)$$

In these formulas, u_B is the value of u at the boundary point B; it is defined as the intersection of the boundary curve with the line β=constant.

Finally for derivatives $\partial^2_{\alpha\beta}$ centering is not possible; however a first order approximation is sufficient. For example $u_{\alpha\beta}$ at point (α_i, β_j) could be approximated by

$$u_{\alpha\beta} \sim \frac{\dfrac{u_{i,j} - u_{i-1,j}}{\Delta\alpha} - \dfrac{u_{i,j-1} - u_{i-1,j-1}}{\Delta\alpha}}{\Delta\beta}$$

This formula assumes that the points (i,j), (i-1,j) and (i-1,j-1) are the only interior points available. However, it is not necessary to determine all the interior neighboring points.

In order to simplify the logic required in determing, at point (i,j), which interior neighboring points are available for use in approximating derivatives by difference quotients, it is convenient to partition the search procedure into two phases: a nearest neighbor search for the four nearest neighbors followed by a search for the four remaining neighboring points. Once one of the latter points is found it is used to find the approximation to $u_{\alpha\beta}$ in the double Taylor expansion for u given by

$$u(\alpha + \Delta\alpha, \beta + \Delta\beta) = u(\alpha,\beta) + \Delta\alpha u_\alpha + \Delta\beta u_\beta + \frac{1}{2}(\Delta\alpha^2 u_{\alpha\alpha} + 2\Delta\alpha\Delta\beta\, u_{\alpha\beta} + \Delta\beta^2 u_{\beta\beta}) \qquad (4.b.26)$$

During the first phase of the boundary search about the point where $u(\alpha,\beta)$ is interior to the domain of integration u_α, u_β, $u_{\alpha\alpha}$ and $u_{\beta\beta}$ are computed. Once it is determined that $u(\alpha+\Delta\alpha, +\Delta\beta)$ is an interior point (here $\Delta\alpha, \Delta\beta$ can be either positive or negative) $u_{\alpha\beta}$ is computed from Equation (4.b.26).

c) Too Near Points

There are a set of interior points for which neither the two step nor the one step difference operators can stably produce an updated solution for the vector w. All such points lie within a thin annular region whose boundaries are the boundary of the domain and a boundary essentially parallel to the boundary of the domain a distance on the order of one third $\Delta\alpha$ or $\Delta\beta$. All points lying within this band take values interpolated between interior and boundary data.

V. Treatment of the Boundary

a) Lagrangian Representation of the Boundary

The boundary of each domain is considered to be represented by a polygon whose vertices are Lagrangian points moving with the local material velocity. At each point i the differential equations

$$\frac{dz_i}{dt} = u_i \quad , \quad \frac{dr_i}{dt} = v_i \qquad i = 1,2,\ldots,n \qquad (5.a.1)$$

are solved for the coordinate pair (z_i, r_i) given the velocity vector $\binom{u_i}{v_i}$ at the point i. In computational space, the α-β plane, Equation (5.a.1) becomes

$$\frac{d\alpha_i}{dt} = u_i\alpha_z + v_i\alpha_r$$

$$i = 1,2,\ldots,n \qquad (5.a.1')$$

$$\frac{d\beta_i}{dt} = u_i\beta_z + v_i\beta_r$$

The values of velocity at each point on the boundary is known at the initial time so that the new boundary position is computed to be the polygon with vertices

$$\alpha_i(t+\Delta t) = \alpha_i(t) + \Delta t(u_i(t)\alpha_z + v_i(t)\alpha_r)$$

$$(5.a.2)$$

$$\beta_i(t+\Delta t) = \beta_i(t) + \Delta t(u_i(t)\ \beta_z + v_i(t)\beta_r)$$

The updated values of the velocity components at the new boundary position is obtained by a space-time extrapolation from the interior. The data chosen for extrapolation to the k^{th} tracer particle is the nearest interior neighbor of the k^{th} point with coordinates $(\alpha_k(t+\Delta t),\ \beta_k(t+\Delta t))$, i.e.

$$u_k(\alpha_k(t+\Delta t),\ \beta_k(t+\Delta t)) = u(\alpha_{i,}^{(t)}\beta_j^{(t)})$$

where α_i and β_j are chosen so that

$$(\alpha_k(t+\Delta t) - \alpha_i(t))^2 + (\beta_k(t+\Delta t) - \beta_j(t))^2$$

is a minimum over all i and j; the values of i and j are mesh crossings interior to the domain of interest.

It is possible to correct the boundary position using the latest values of the velocity components by using the corrector formula

$$\alpha_i(t+\Delta t) = \alpha_i(t) + \frac{\Delta t}{2} (\tilde{u}(t) + \tilde{u}(t+\Delta t))$$

$$\beta_i(t+\Delta t) = \beta_i(t) + \frac{\Delta t}{2} (\tilde{v}(t) + \tilde{v}(t+\Delta t)) \tag{5.a.3}$$

where $\tilde{u}$ and $\tilde{v}$ are the transformed velocity components in the α-β plane; they are given by the right hand side of Equation (5.a.1'). It has been found that in the numerical experiments carried out thus far, very little difference appears in the solution of the position of the boundary. This is probably due to the fact that the time step is very small being based upon the relatively high sound speed found in elastic materials. On the other hand, boundary velocities are usually much smaller than the characteristic sound speeds so that, since the nearest neighbor is usually unchanged for the k^{th} boundary point, the formula (5.a.2) is sufficient.

The boundary is moved by the integration of the differential equations (5.a.1'), the integrands being obtained by extrapolation from interior data. As this boundary sweeps through the Eulerian mesh data must be defined at points on the boundary which coincide with the Eulerian mesh lines. This set of points is used to augment the set of interior points when difference approximations to partial derivatives in the alpha and beta direction are computed (see Equations (4.b.24) and (4.b.25)).

b) Free Surface Boundary Conditions

The boundary of any domain is composed of segments which constitute one side of a slip line, i.e., an interface, or a free surface. In the later case, a purely hydrodynamic code only requires the vanishing of the pressure,

$$p = 0$$

as a boundary condition. However, for elastic domains, the pressure is just one component of the stress. The proper condition is that the normal stress must vanish:

$$\tau_n = 0 \tag{5.b.1}$$

In order to compute τ_n we let $\tan\psi$ measure the local slope of the boundary. Let $\hat{n}$ represent the local unit normal vector to the boundary

$$\hat{n} = \begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix} \qquad (5.b.2)$$

and $\hat{t}$ represent the local unit tangent to the boundary

$$\hat{t} = \begin{pmatrix} -\cos\psi \\ \sin\psi \end{pmatrix} \qquad (5.b.3)$$

Let the stress matrix T be given by

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \qquad (5.b.4)$$

Then the normal stress in the normal direction is

$$\begin{aligned} T\hat{n}\cdot\hat{n} &= \tau_{11} \sin^2\psi + 2\tau_{12} \sin\psi \cos\psi \\ &\quad + \tau_{22} \cos^2\psi = 0 \end{aligned} \qquad (5.b.5)$$

Here $\tau_{ij} = S_{ij} - p\delta_{ij}$.

Equation (5.b.5), for a hydrodynamic material reduces to p=0.

The tangential stress in the tangential direction

$$T\hat{t}\cdot\hat{t} = \tau_{11} \sin^2\psi - 2\tau_{12} \sin\psi \cos\psi + \tau_{22} \cos^2\psi \qquad (5.b.6)$$

may be arbitrary.

In addition the tangential component of the normal stress vanishes

$$\begin{aligned} T\hat{n}\cdot t &= (\tau_{22} - \tau_{11})\sin\psi \cos\psi - \tau_{12}(\cos^2\psi - \sin^2\psi) \\ &= 0 \end{aligned} \qquad (5.b.7)$$

Under conditions (3.b.5), (5.b.6) and (5.b.7) we find that

$$\tau_{11} = T\hat{t}\cdot\hat{t}\cos^2\psi$$

$$\tau_{12} = -T\hat{t}\cdot\hat{t}\sin\psi\cos\psi \qquad (5.b.8)$$

$$\tau_{22} = T\hat{t}\cdot\hat{t}\sin^2\psi$$

are the desired values of the components of the stress tensor in terms of the extrapolated values of the tensor T obtained from the interior of the domain.

c) Interface Boundary Conditions

The conditions to be applied to the boundary at the interface are simple generalizations of the three conditions described in the above section.

First there is a kinematic condition which states that the jump in the normal velocity at the interface vanishes, i.e.

$$(u^{(1)} - u^{(2)})\sin\psi + (v^{(1)} - v^{(2)})\cos\psi = 0 \qquad (5.c.1)$$

Here we have used superscripts to denote the material number on each side of the interface. Condition (5.c.1) is just the algebraic counterpart of the jump condition $[\vec{u}\cdot\hat{n}] = 0$ prescribed at the interface.

A condition on the stress at the interface, is that $[T\hat{n}\cdot\hat{n}] = 0$ i.e.,

$$\left(\frac{S_{22}^{(1)} + S_{11}^{(1)}}{2} - p^{(1)} - \frac{S_{22}^{(2)} + S_{11}^{(2)}}{2} + p^{(2)}\right)$$

$$+ \left(\frac{S_{22}^{(1)} - S_{11}^{(1)}}{2} - \frac{S_{22}^{(2)} - S_{11}^{(2)}}{2}\right)(\cos^2\psi - \sin^2\psi) \qquad (5.c.2)$$

$$+ 2(S_{12}^{(1)} - S_{12}^{(2)})\sin\psi\cos\psi = 0$$

In addition on each side of the interface $T\hat{n}\cdot\hat{t} = 0$, i.e.,

$$(S_{22}^{(1)} - S_{11}^{(1)}) \sin\psi \cos\psi - S_{12}^{(1)} (\cos^2\psi - \sin^2\psi)$$

$$= 0 \qquad (5.c.3)$$

$$(S_{22}^{(2)} - S_{11}^{(2)}) \sin\psi \cos\psi - S_{12}^{(2)} (\cos^2\psi - \sin^2\psi)$$

$$= 0 \qquad (5.c.4)$$

The quantity $T\hat{n}\cdot\hat{n}$ is computed from each side of the interface. In order to assure that the normal velocity remains continuous we introduce a new value of $T\hat{n}\cdot\hat{n}$ in terms of these computed value via

$$T\hat{n}\cdot\hat{n} = \frac{\rho^{(1)}\ T\hat{n}\cdot\hat{n}^{(2)} + \rho^{(2)}\ T\hat{n}\cdot\hat{n}^{(1)}}{\rho^{(1)} + \rho^{(2)}} \qquad (5.c.5)$$

A value of $T\hat{n}\cdot\hat{t}$ is formed via the welded boundary condition assumption

$$T\hat{n}\cdot\hat{t}^{(1)} = T\hat{n}\cdot\hat{t}^{(2)} = 1/2(T\hat{n}\cdot\hat{t}^{(1)} + T\hat{n}\cdot\hat{t}^{(2)}) \qquad (5.c.6)$$

The value of $T\hat{t}\cdot\hat{t}$ is obtained from the interior of each side of the interface, i.e. both $T\hat{t}\cdot\hat{t}^{(1)}$, $T\hat{t}\cdot\hat{t}^{(2)}$ are extrapolated along almost characteristic directions from the corresponding interior of each material.

We now have the fact that given $T\hat{n}\cdot\hat{n}$ from Equation (5.c.5) and $T\hat{n}\cdot\hat{t}$ from the welded boundary condition (5.c.6) as well as the extrapolated values of $T\hat{t}\cdot\hat{t}$ from each side one can compute the stress components from the set of linear equations

$$T\hat{n}\cdot\hat{n} + p = 2\sin\psi \cos\psi\ S_{12} + \sin^2\psi\ S_{11} + \cos^2\psi\ S_{22}$$

$$T\hat{n}\cdot\hat{t} = -(\cos^2\psi - \sin^2\psi) S_{12} - \sin\psi \cos\psi\ S_{11}$$

$$- \sin\psi \cos\psi\ S_{22}$$

$$T\hat{t}\cdot\hat{t} + p = -2\sin\psi \cos\psi\ S_{12} + \cos^2\psi\ S_{11} + \sin^2\psi\ S_{22}$$

The determinent of the above system is unity.

By applying Kramer's rule we find

$$S_{11} = T\hat{n}\cdot\hat{n}\,\sin^2\psi + T\hat{t}\cdot\hat{t}\,\cos^2\psi - 2T\hat{n}\cdot\hat{t}\,\sin\psi\,\cos\psi + p \qquad (5.c.7)$$

$$S_{12} = (\sin^2\psi - \cos^2\psi)\,T\hat{n}\cdot\hat{t} + (T\hat{n}\cdot\hat{n} - T\hat{t}\cdot\hat{t})\,\sin\psi\,\cos\psi \qquad (5.c.8)$$

$$S_{22} = T\hat{n}\cdot\hat{n}\,\cos^2\psi + 2T\hat{n}\cdot\hat{t}\,\sin\psi\,\cos\psi + T\hat{t}\cdot\hat{t}\,\sin^2\psi + p \qquad (5.c.9)$$

The condition on the normal velocity of the interface $u_\perp$ is taken to be

$$u_\perp = (\rho^{(1)}\,u^{(1)} + \rho^{(2)}\,u^{(2)})/(\rho^{(1)} + \rho^{(2)}) \qquad (5.c.10)$$

while the tangential component $u_\parallel^{(1)}$, $u_\parallel^{(2)}$ are extrapolated from the corresponding interior positions along almost characteristic directions.

d) Characteristic Equations

We have carried out a formulation of the equations of motion at interface boundaries in characteristic form. It is desirable to formulate the difference problem on a coordinate system in which the two coordinates are locally orthogonal and parallel to the boundary. Since the boundary will exhibit a spatial variation in slope which will change in time, it is convenient to recast the basic differential equations in a form where differentiation is automatically carried out in a direction more or less normal and parallel to the boundary. The corresponding difference scheme thus generated can be aligned with the boundary - differences being taken perpendicular and parallel to the boundary.

One first takes an appropriate linear combination of the momentum equations and constitutive relations, i.e.,

$$\begin{aligned}
&\rho\frac{Du}{Dt} + p_z - \left(S_{11,z} + S_{12,r} + \frac{S_{12}}{r}\right) \\
&\quad - \tan\theta\left[\rho\frac{Dv}{Dt} + p_r - \left(S_{12,z} + S_{22,r} + \frac{2S_{22}+S_{11}}{r}\right)\right] \\
&\quad - \frac{\sin\theta}{\sqrt{\mu/\rho}}\left[\frac{DS_{11}}{Dt} - \frac{2\mu}{3}\left(2u_z - v_r - \frac{v}{r}\right)\right] \\
&\quad - \frac{\cos 2\theta}{\sqrt{\mu/\rho}\,\cos\theta}\left[\frac{DS_{12}}{Dt} - \mu(u_r + v_z)\right] \\
&\quad + \frac{\sin\theta}{\sqrt{\mu/\rho}}\left[\frac{DS_{22}}{Dt} - \frac{2\mu}{3}\left(2v_r - u_z - \frac{v}{r}\right)\right] = 0
\end{aligned} \qquad (5.d.1)$$

Then by introducing the definition of characteristic differentiation in terms of the particle derivative via

$$\frac{d}{dt} = \frac{D}{Dt} + \left(\frac{\mu}{\rho}\right)^{1/2}\sin\theta\,\frac{\partial}{\partial z} + \left(\frac{\mu}{\rho}\right)^{1/2}\cos\theta\,\frac{\partial}{\partial r}$$

$$= \frac{\partial}{\partial t} + \left(u + \left(\frac{\mu}{\rho}\right)^{1/2}\sin\theta\right)\frac{\partial}{\partial z} + \left(v + \left(\frac{\mu}{\rho}\right)^{1/2}\cos\theta\right)\frac{\partial}{\partial r} \tag{5.d.2}$$

Equation (5.d.1) can be written as

$$\rho\left[\frac{du}{dt}\cos\theta - \frac{dv}{dt}\sin\theta\right] - \frac{\sin 2\theta}{\sqrt{\mu/\rho}}\left[\frac{dS_{11}}{dt} - \frac{dS_{22}}{dt}\right] - \frac{\cos 2\theta}{\sqrt{\mu/\rho}}\frac{dS_{12}}{dt}$$

$$= \sqrt{\mu/\rho}\,\frac{d}{d\xi}(u\sin\theta - v\cos\theta) + \frac{d}{d\xi}(S_{11}\cos^2\theta + S_{22}\sin^2\theta$$

$$- p - 2S_{12}\sin\theta\cos\theta) + \frac{1}{r}(S_{12}\cos\theta - (2S_{22} + S_{11})\sin\theta) \tag{5.d.3}$$

This is the characteristic equation along the conoid with the local disturbances propagate with speed $C_s^2 = \mu/\rho$, the shear speed, relative to the particle speed. The angle θ is measured from the image of the characteristic on the plane t=0 from the r axis and the directional derivative $d/d\xi$, defined in the plane t=0 and directed tangent to the base of the conoid, is defined by

$$\frac{d}{d\xi} = \cos\theta\,\frac{\partial}{\partial z} - \sin\theta\,\frac{\partial}{\partial r} \tag{5.d.4}$$

In addition to the shear conoid, there appears another characteristic cone defined from the characteristic speed $C_d^2 = C^2 + 4\mu/3\rho$, C being the sound speed. Obviously, disturbances which propagate at speed C_d relative to the material particle speed will travel further in time Δt than disturbances which travel at the shear speed C_s. Hence, for wave motion in an elastic medium, the shear conoid lies inside the sonic conoid.

In order to find the characteristic equations along the sonic cone we again take linear combinations of the continuity and momentum equations, compatability equations and the particle path equation which relates changes in pressure to density on a particle. The result is

$$\frac{D\rho}{Dt} + \rho(u_z + v_r + \frac{v}{r}) + \frac{\sin\theta}{C}\sqrt{1 + \frac{4\mu}{3\rho C^2}}\left[\rho\frac{du}{dt} + p_z\right.$$

$$- (S_{11,z} + S_{12,r} + \left.\frac{S_{12}}{r})\right] + \frac{\cos\theta}{C}\sqrt{1 + \frac{4\mu}{3\rho C^2}}\left[\rho\frac{dv}{dt} + p_r\right.$$

$$- (S_{12,z} + S_{22,r} \left.\frac{2S_{22} + S_{11}}{r})\right] + \frac{1}{C^2}\left[\frac{Dp}{Dt} - C^2\frac{D\rho}{Dt}\right]$$

$$- \frac{\sin 2\theta}{C^2}\left[\frac{DS_{11}}{Dt} - \frac{2\mu}{3}(2u_z - v_r - \frac{v}{r})\right] \qquad (5.d.5)$$

$$- \frac{2\sin\theta\cos\theta}{C^2}\left[\frac{DS_{12}}{Dt} - \mu(u_r + v_z)\right] - \frac{\cos 2\theta}{C^2}\left[\frac{DS_{22}}{Dt}\right.$$

$$\left.- \frac{2\mu}{3}(2v_r - u_z - \frac{v}{r})\right] = 0$$

Equation (5.d.5) can, after a fair amount of algebraic manipulation is performed, be put into the characteristic form

$$\rho C\sqrt{1 + \frac{4\mu}{3\rho C^2}}\left[\frac{du}{dt}\sin\theta - \frac{dv}{dt}\cos\theta\right] + \frac{dp}{dt}$$

$$- \sin^2\theta\frac{dS_{11}}{dt} - 2\sin\theta\cos\theta\frac{dS_{12}}{dt} - \cos^2\theta\frac{dS_{22}}{dt}$$

$$= (\frac{2\mu}{3} - \rho C^2)\frac{d}{d\xi}\left[u\cos\theta - v\sin\theta\right] \qquad (5.d.6)$$

$$+ C\sqrt{1 + \frac{4\mu}{3\rho C^2}}\frac{d}{d\xi}(\sin\theta\cos\theta(S_{11} - S_{22})$$

$$+ (\cos^2\theta - \sin^2\theta)S_{12}) + \left\{(C\sqrt{1 + \frac{4\mu}{3\rho C^2}}\left[S_{12}\sin\theta\right.\right.$$

$$\left.\left.+ (2S_{22} + S_{11})\cos\theta\right] + (\frac{2\mu}{3} - \rho C^2)v\right\}/r$$

We have used the following definitions in Equation (5.d.6). The derivative in the bicharacteristic direction is

$$\frac{d}{dt} = \frac{D}{Dt} + C\sqrt{1 + \frac{4\mu}{3\rho C^2}}(\frac{\partial}{\partial z}\sin\theta + \frac{\partial}{\partial r}\cos\theta) \qquad (5.d.7)$$

while the material particle derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial z} + v \frac{\partial}{\partial r} \qquad (5.d.C)$$

As before Equation (5.d.4) def... the directional derivative $d/d\xi$ which is used in Equation (5.d.)

The method of computation is straight forward; the derivatives appearing in the above equations are replaced by differences in the direction in which the derivatives were defined. For example, Equation (5.d.3) can be written in finite difference form as

$$\begin{aligned}
&\sqrt{\rho_0^{(1)}\mu^{(1)}} \left(\cos\theta_1^{(1)} u^{(1)} - \sin\theta_1^{(1)} v^{(1)}\right) \\
&\quad - 2 \sin\theta_1^{(1)} \cos\theta_1^{(1)} (S_{11}^{(1)} - S_{22}^{(1)}) \\
&\quad - (\cos^2\theta_1^{(1)} - \sin^2\theta_1^{(1)}) S_{12}^{(1)} \qquad (5.d.9) \\
&\quad = \sqrt{\rho_0^{(1)}\mu^{(1)}} \left(\cos\theta_1^{(1)} u^{(0)} - \sin\theta_1^{(1)} v^{(0)}\right) \\
&\quad - 2 \sin\theta_1^{(1)} \cos\theta_1^{(1)} (S_{11}^{(0)} - S_{22}^{(0)}) \\
&\quad - (\cos^2\theta_1^{(1)} - \sin^2\theta_1^{(1)}) S_{12}^{(0)} \\
&\quad + \Delta t. \ \text{(difference approximations to the RHS of Equation (5.d.3))}
\end{aligned}$$

The differencing is along a generatrix of the shear cone and along a line perpendicular to the generatrix in the z-r plane. This point of intersection is denoted by the superscript zero while the vertex of the shear cone is denoted by superscript one. The subscript one denotes the particular angle theta chosen to define the ray for the integration; the superscript on the angle θ denotes the material number. Equation (5.d.9) as it now stands has been written on one side of the interface and therefore a similar relation must be written for the other side. Thus, whenever we have a term $\theta_1^{(1)}$ it is replaced by $\theta_1^{(2)}$, $\rho_0^{(1)}$ is replaced by $\rho_0^{(2)}$, $\mu^{(1)}$ is replaced by $\mu^{(2)}$ and $u^{(1)}$ and $v^{(1)}$ are understood to be the velocity components at the vertex of the conoid on the other side of the interface, i.e. $u^{(2)}$ and $v^{(2)}$. This gives us a second algebraic equation representing integration along the shear cone to the interface from the second material.

The same procedure is now applied to Equation (5.d.6) with the exception that two angles along the dialatational conoid are chosen. In general these angles are distinct from those chosen for the shear conoid. This gives us four more algebraic relations for the twelve unknowns $u^{(1)}$, $v^{(1)}$, $p^{(1)}$, $s_{11}^{(1)}$, $s_{12}^{(1)}$, $s_{22}^{(1)}$ and $u^{(2)}$, $v^{(2)}$, $p^{(2)}$, $s_{11}^{(2)}$, $s_{12}^{(2)}$, $s_{22}^{(2)}$.

One now looks at the basic stress strain relations which include the terms accounting for rotation:

$$s_{11,t} = -(u\, s_{11,z} + s_{12}\, v_z - \frac{4}{3}\mu u_z) - (v\, s_{11,r} - s_{12}\, u_r + \frac{2\mu}{3} v_r) - \frac{2\mu}{3}\frac{v}{r} \qquad (5.d.10a)$$

$$s_{12,t} = -(u\, s_{12,z} - (\mu + \frac{s_{11}-s_{22}}{2})\, v_z - (v\, s_{12,r} - (\mu - \frac{s_{11}-s_{22}}{2})\, u_r) \qquad (5.d.10b)$$

$$s_{22,t} = -(u\, s_{22,z} - s_{12}\, v_z + \frac{2\mu}{3} u_z) - (v\, s_{22,r} + s_{12}\, u_r - \frac{4}{3}\mu\, v_r) - \frac{2\mu}{3}\frac{v}{r} \qquad (5.d.10c)$$

If we neglect these rotation terms and multiply Equation (5.d.10a) and Equation (5.d.10c) by 3/2 they may be added to obtain

$$\frac{3}{2}\frac{d}{dt}(s_{11} + s_{22}) = \mu(u_z + v_r - \frac{2v}{r}) \qquad (5.d.11)$$

Equation (5.d.10b) may now be added and subtracted to Equation (5.d.11) to yield

$$\frac{d}{dt}\left[\frac{3}{2}(s_{11} + s_{22}) \pm s_{12}\right] = \mu\left[(\frac{\partial}{\partial z} \pm \frac{\partial}{\partial r})(u \pm v) - \frac{2v}{r}\right] \qquad (5.d.12)$$

Equation (5.d.12) has a directional derivative, $\frac{d}{dt}$, defined along the particle path while the right hand side is expressed in terms of a fixed directional derivative, i.e. fixed along the directions $\pm$ 45 degrees. Integration of Equation (5.d.12) along the particle path on each side of the interface gives us two more independent algebraic relations for the twelve unknowns. The remaining four relations are obtained from the boundary conditions.

If the slope of the interface is $\tan\psi = dz/dr$ then the continuity of normal velocity yields

$$(u^{(1)} - u^{(2)}) \sin\psi + (v^{(1)} - v^{(2)}) \cos\psi = 0 \qquad (5.d.13)$$

The remaining conditions are obtained from constraints on the stress tensor T:

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \qquad (5.d.14)$$

Let $\hat{n} = \begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix}$ be the unit normal to the interface with slope $\tan\psi$; therefore the unit tangent is $\hat{t} = \begin{pmatrix} -\cos\psi \\ \sin\psi \end{pmatrix}$. Continuity of the normal stress, $[T\hat{n}\cdot\hat{n}] = 0$ yields our second boundary condition, Equation (5.c.2).

The remaining two conditions can be obtained by placing a condition on the components of the normal stress, $T\hat{n}$, in the tangential direction, $\hat{t}$. We again invoked the condition $T\hat{n}\cdot\hat{t} = 0$ on side 1 and side 2.

Hence

$$-(S_{11}^{(1)} - S_{22}^{(1)}) \sin\psi \cos\psi - S_{12}^{(1)} (\cos^2\psi - \sin^2\psi) = 0 \qquad (5.d.15)$$

$$-(S_{11}^{(2)} - S_{22}^{(2)}) \sin\psi \cos\psi - S_{12}^{(2)} (\cos^2\psi - \sin^2\psi) = 0 \qquad (5.d.16)$$

are the required conditions.

The entire system of equations can be written in the following form

$$A\,w = b \qquad (5.d.17)$$

where the unknown vector w has entries given by

$$
w = \begin{pmatrix} u^{(1)} \\ v^{(1)} \\ p^{(1)} \\ s_{11}^{(1)} + s_{22}^{(1)} \\ s_{11}^{(1)} - s_{22}^{(1)} \\ s_{12}^{(1)} \\ u^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ s_{12}^{(2)} \end{pmatrix}
$$

while the coefficient matrix A of order 12 has row entries defined by

1) Continuity of normal velocity, Equation (5.d.13)

2) Continuity of normal stress, Equation (5.c.2)

3) Boundary condition, Equation (5.d.15)

4) Boundary condition, Equation (5.d.16)

5) Difference equations, Shear conoid material 1, Equation (5.d.9)

6) Difference equations, Shear conoid material 2

7) Difference equation, Dialatational conoid-material 1 with angle $\theta_1^{(1)}$, (difference form of Equation (5.d.6)

8) Difference equation, sonic conoid-material 1 with angle $\theta_2^{(1)}$

9) Difference equation, sonic conoid-material 2, with angle $\theta_2^{(2)}$

10) Difference equation, sonic conoid-material 2, with angle $\theta_2^{(2)}$

11) Difference equation, Particle path relationship for material 1, Equation (5.d.12) in difference form

12) Difference equation, Particle path relationship for material 2, Equation (5.d.12) in difference form

In the above listing the entries actually used in row 8 and row 10 are the differences of the two difference equations defined by integration along the angles $\theta_1^{(i)}$, $\theta_2^{(i)}$ for each of the materials, i=1,2.

The above formulation has been programmed and is presently undergoing testing. It is considered to be a prototype of methods which may be used to predict interactions of two dissimilar materials undergoing elastic impact.

VI. Results

In order to demonstrate the versatility of the algorithm, two problems were considered. The first problem consists of a penetrator composed of a 90/25 tungsten alloy impacting at 0.142 cm/μsec upon a one inch thick RHA plate. The configuration at moment of impact is shown in Figure (1). The second problem, shown in Figure (2), is a similar tungsten projectile but now enclosed in a Maraging-300 steel sheath. The impact velocity is the same as in the first problem. Each material in both problems is assumed to have an equation of state which is given by Tillotson:

Compressed states: $p = p_c$

$$p_c = \left(a + \frac{b}{\frac{E}{E_0\eta^2}+1}\right)E\rho + A\xi + B\xi^2 \qquad \rho > \rho_0,\quad 0 < E < E_s$$

where $\xi = \eta - 1$, $\eta = \rho/\rho_0$

Expanded States: $p = p_e$

$$p_e = aE\rho + \left(\frac{bE\rho}{\frac{E}{E_0\eta^2}+1} + A\xi e^{-\tau\left(\frac{1}{\eta}-1\right)}\right)e^{-\alpha\left(\frac{1}{\eta}-1\right)^2} \qquad \rho < \rho_0,\quad E > E_s$$

Intermediate States:

$$p = \frac{(E - E_s)p_e + (E_s' - E)p_c}{E_s' - E_s} \qquad \rho < \rho_0,\quad E_s < E < E_s'$$

The constants used in the above equation of state for the present calculation are given in the following table.

	ρ_0	a	b	A	B	α	z	E_0	E_s	E'_s	μ	Y_c
	gm/cm^3			10^{12}dynes/cm^2	10^{12}dynes/cm^2			10^{12}ergs/gm	10^{12}ergs/gm	10^{12}ergs/gm	10^{12}dynes/cm^2	10^{12} dynes/cm^2
90-25 Tungsten Alloy	17.04	0.5	1.04	3.08	2.5	10.	10.	0.225	0.0111	0.056	1.2967	0.0111
RHA	7.8	0.5	1.50	1.28	1.05	5.	5.	0.095	0.0244	0.102	0.8077	0.01219
Maraging 300	8.0	0.5	1.50	1.28	1.05	5.	5.	0.095	0.0244	0.102	0.796	0.0192

The coefficient of viscosity K in Equation (4.a.7) was taken to be 1.8. The two transformations (2.1) and (2.2), which are presently coded inline rather than in a function subprogram form, were used in each material. The constants for the transformations applied to each material domain are given in the accompanying table.

	L	Transformation (2.1) $a=\frac{r_{max}}{L-1/2}$	d	Transformation (2.2) D	q
90/25 Tungsten	17	1.75/16.5	1.75	Δz^{-1}	0
RHA	24	9/23.5	0.5	Δz^{-1}	0
Maraging-300	17	1.75/16.5	1.75	Δz^{-1}	0

The constants used in the transformations for the projectile were chosen with $d \geq r_{max}$ so that uniform spacing in r is achieved. For the target, fine spacing in the region of impact is desired so $d < r_{max}$.

The mesh $(i\Delta\alpha, j\Delta\beta)$, which was chosen such that $1 \leq i \leq I$, $1 \leq j \leq J$, is given in the following table for each of the domains.

	I	J	i_0-right	j_0	i_0-left
90/25 Tungsten	121	20	120	17	21
RHA	31	25	25	24	16
Maraging-300	61	20	60	17	11

Here i_0 and j_0 represent respectively the initial α position of the material right justified on the mesh and the initial maximum height of the material in the β direction. The initial starting value of i_0 left justified is also shown. The minimum value of j_0 is 1 for all domains.

The final configuration after 30 microseconds for the unsheathed penetrator is shown in Figure 3. The letter P appears at each mesh point when the material is at a stress level which satisfies the yield condition. At such points the material behaves plastically. Figure 4 corresponds to the final state of the sheathed penetrator at 30 microseconds. At this time there is slightly less deformation of the nose of this penetrator compared to the unsheathed penetrator. Because of the velocity and density ratio between the penetrator and target, the rod is not significantly deformed during the penetration process except at the nose where some blunting and mushrooming of the initial spherical shape takes place.

In both cases, Figures (3) and (4) we see that the rod has completely penetrated the target and is emerging from the far face. Since spall and other fracture mechanisms have not been incorporated into the present model, plugging is not accounted for; obviously the back face of the target has stretched beyond that which would occur for RHA.

Figure (5) compares the pressure history at the interface on the axis of symmetry of the penetrator for both of the above problems, while Figure (6) compares the pressure history in the target for both problems. The □ symbol is the two material curve while the Δ is the three material curve. For this high velocity, the pressure is continuous across the interface since the stress level is on the order of 1/10 the pressure levels. Thus, on the scale being plotted, both sets of curves, penetrator and target, are nearly coincident. Figure (7) is a plot of the penetration depth D vs time. Over the first 20μ seconds the penetrator leading edge is moving at an average speed of -.06 cm/μsec. It is clear, from this graph, that the speed is increasing since the target is failing beyond this time. Similar behavior is exhibited for both problems. At 30μ sec., the sheath is still moving at -0.142 cm/μsec. as is the penetrator for approximately the last 90% of its length. Hence the residual velocity, although not computed by integration over the volume region of space defining the projectile is on the order of >.9 initial velocity.

Computed stress levels in the sheath at this time are low. The maximum pressure, at the interface near the leading edge, is approximately one kilobar but on the average the pressure lies between one kilobar and one hundred bars. Maximum and average stress levels are similar. The maximum pressure transmitted to the target is somewhat larger for the sheathed rod impacting although the pressure histories are similar. In both cases, the projectile remains compressive near the leading edge out to 25 microseconds.

For the unsheathed penetrator problem, approximately 250 mesh points were used in the targets and 2000 points in the penetrator. This problem ran to 30 microseconds problem time in 2271 seconds on a CDC 6600. For the problem with sheath the target and the penetrator have the same number of points as the first problem, while the sheath contained 1000 mesh points. The computation time for this problem is 2169 seconds.

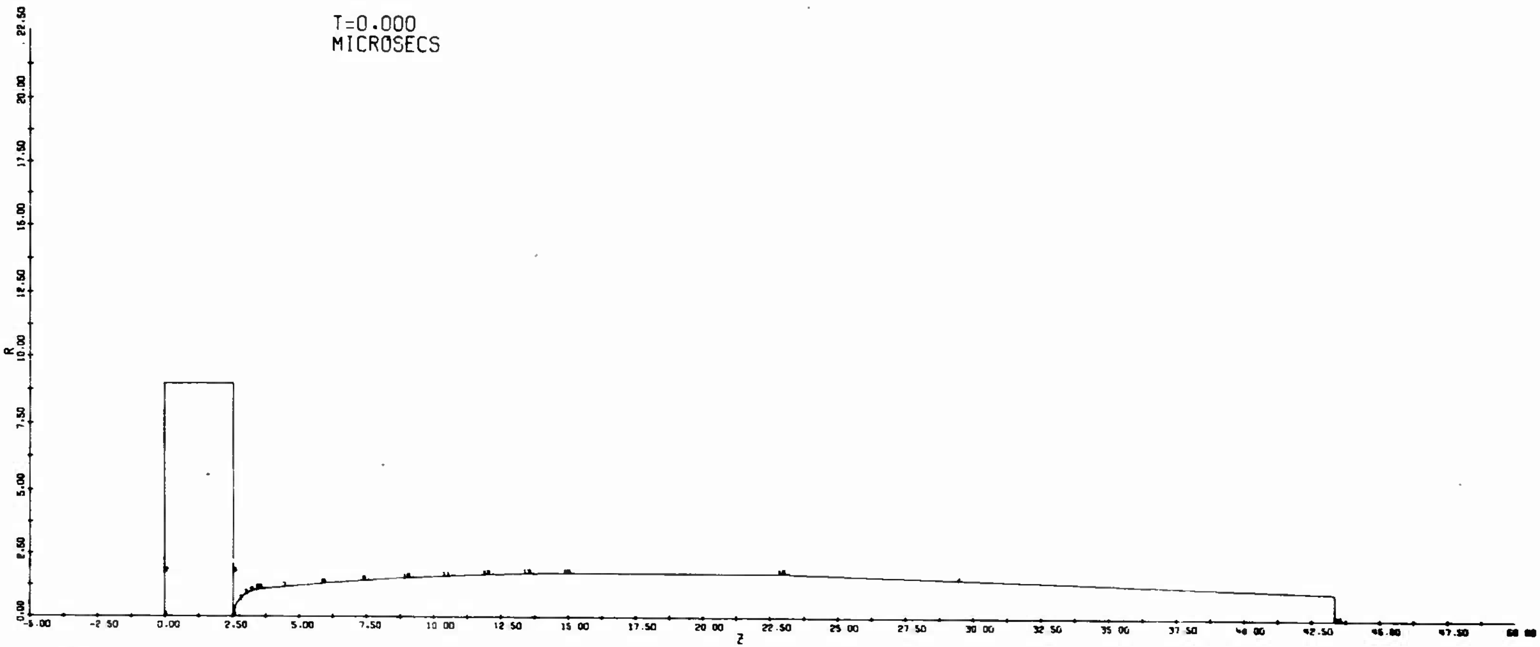

T=0.000
MICROSECS
R
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50
Z
-5.00
-2.50
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50
25.00
27.50
30.00
32.50
35.00
37.50
40.00
42.50
45.00
47.50
50.00

FIGURE 1

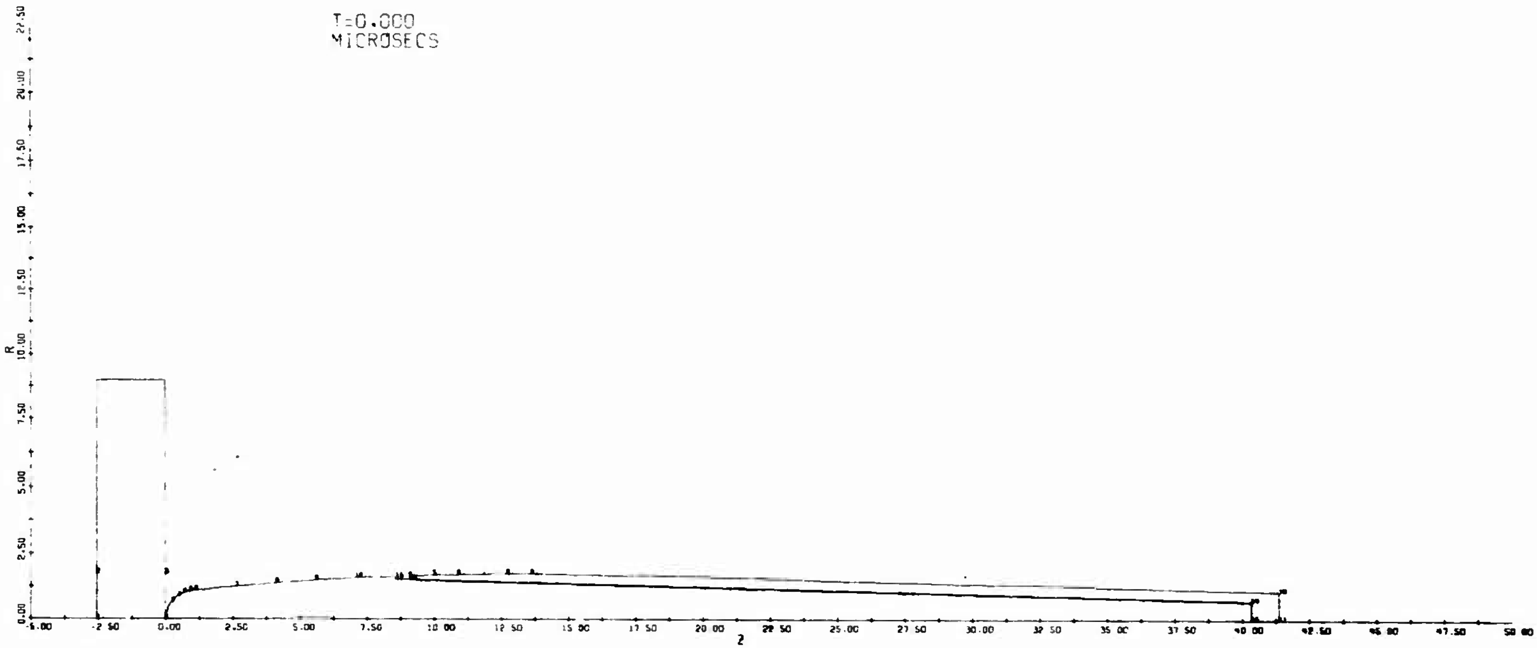


FIGURE 2

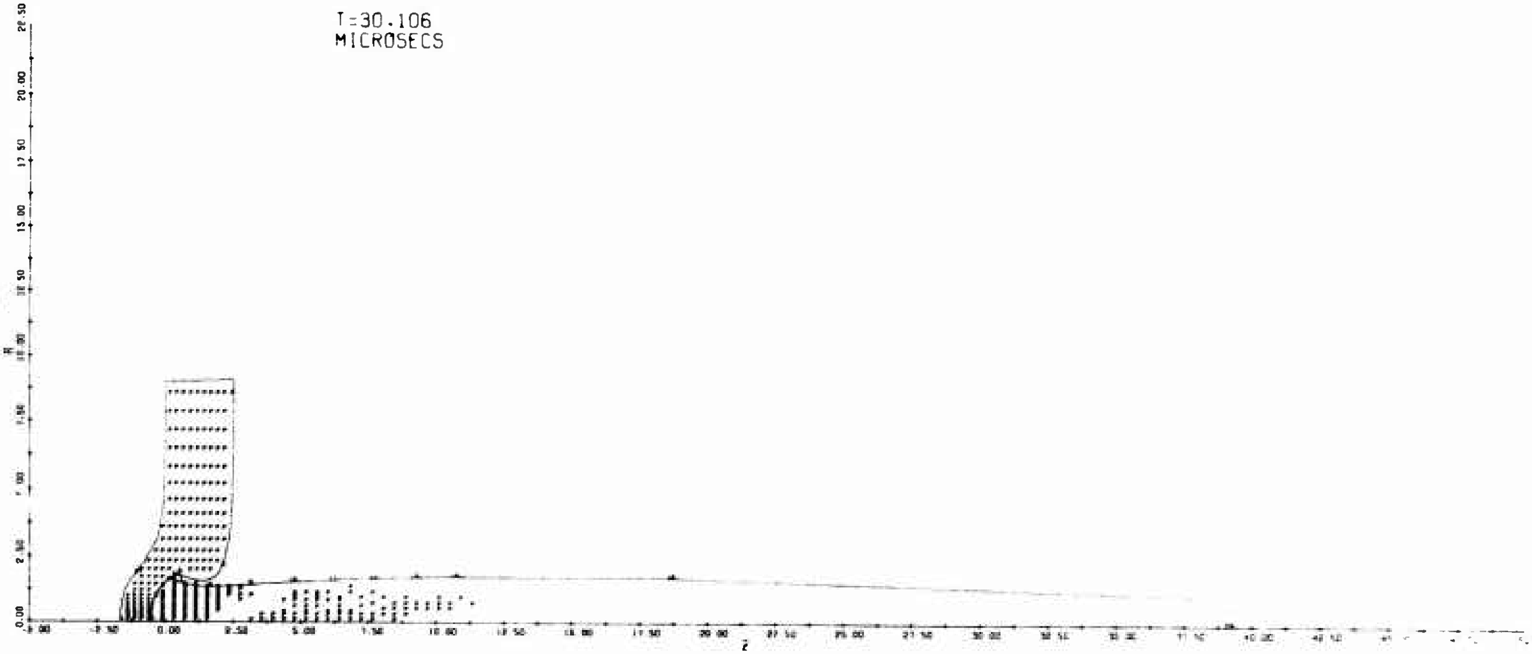
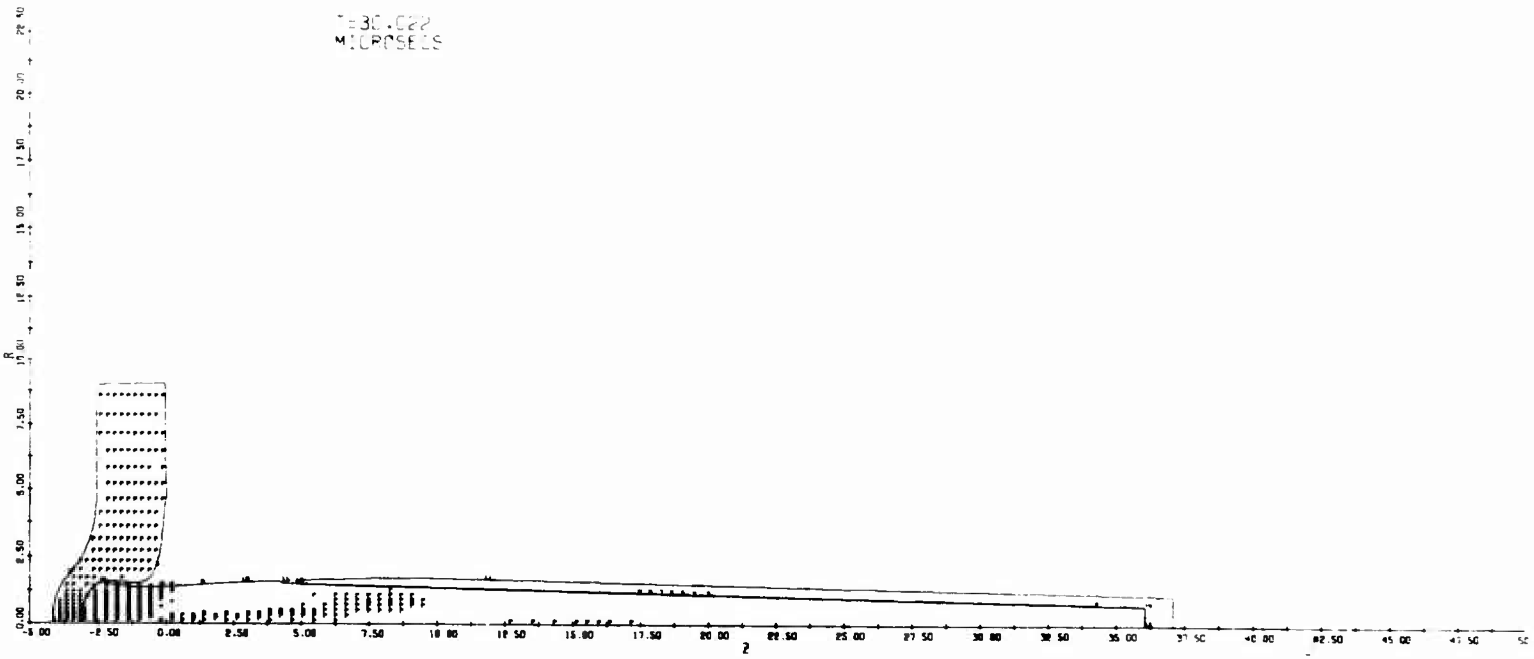
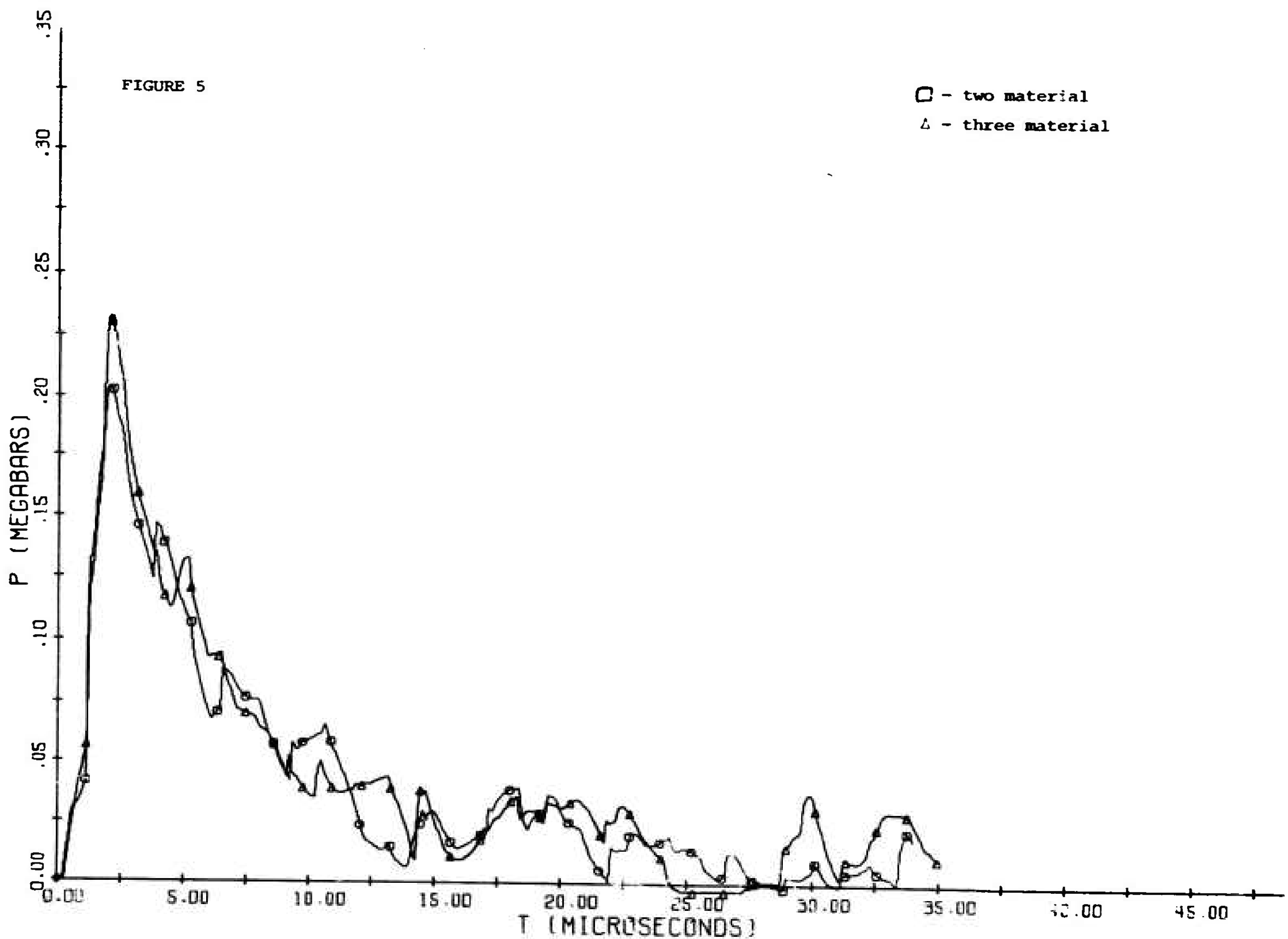


FIGURE 3



T=30.022
MICROSECS
R
Z
-5.00
-2.50
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50
25.00
27.50
30.00
32.50
35.00

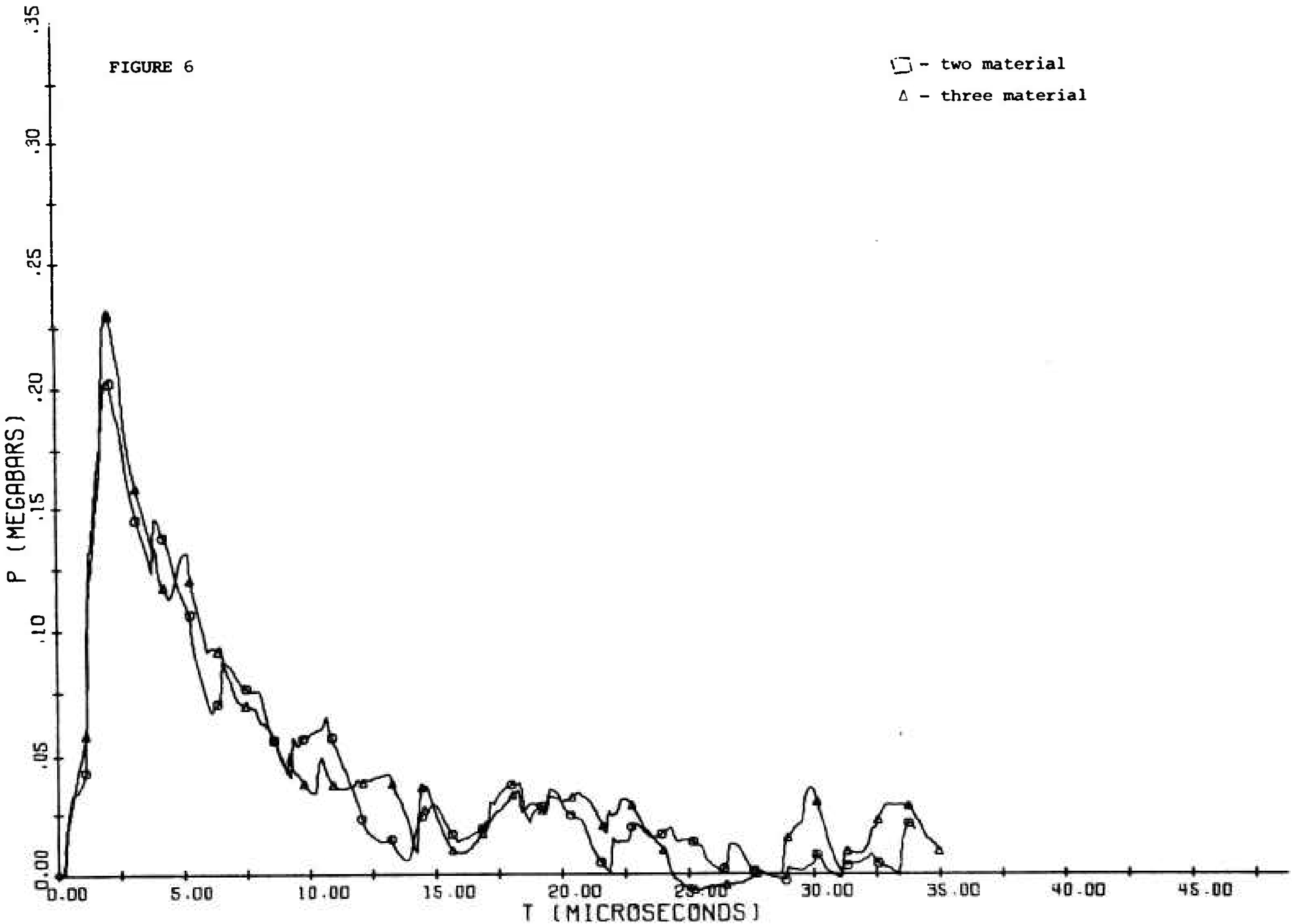
FIGURE 4



- two material
Δ - three material
P (MEGABARS)
.35
.30
.25
.20
.15
.10
.05
0.00
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
45.00
T (MICROSECONDS)

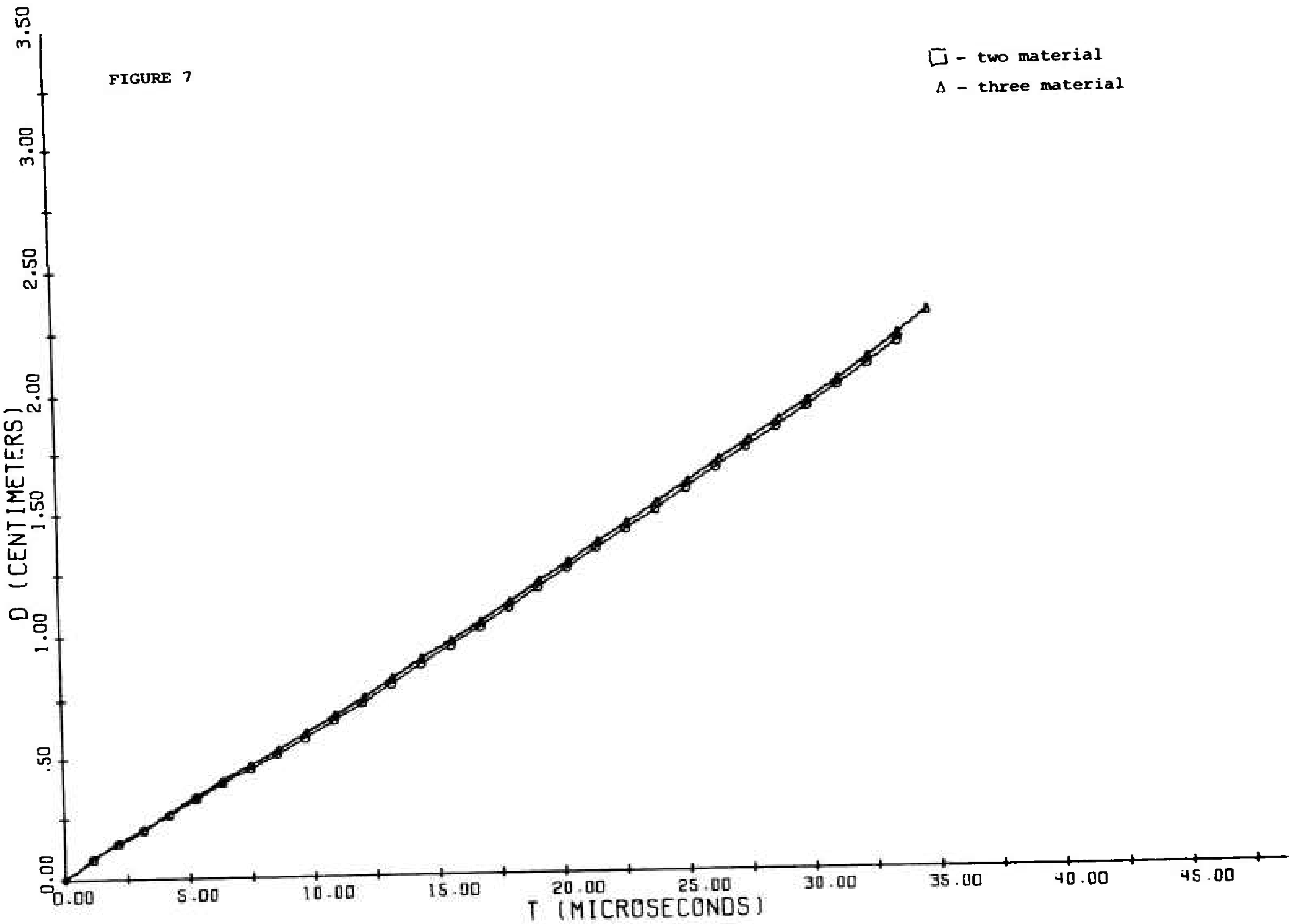
FIGURE 5

FIGURE 6



□ - two material
Δ - three material
P (MEGABARS)
0.00
.05
.10
.15
.20
.25
.30
.35
T (MICROSECONDS)
0.00
5.00
10.00
15.00
20.00
30.00
35.00
40.00
45.00

FIGURE 7



□ - two material
Δ - three material
D (CENTIMETERS)
0.00
.50
1.00
1.50
2.00
2.50
3.00
3.50
T (MICROSECONDS)
0.00
5.00
10.00
15.00
20.00
25.00
30.00
35.00
40.00
45.00

VII. References

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3. Lax, P. D. and Wendroff, B., Difference Schemes for Hyperbolic Equations with Higher Order Accuracy, Comm. Pure and Appl. Math., Vol. 17, pp. 381-98 (1964).